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References

References should appear as footnotes, numbered consecutively, and arranged thus:

[1] A. B. Smith, Phys. Rev. **41**, 852 (1932).

[2] G. R. Harrison, *Atoms in Action* (William Morrow & Company, New York, 1939), first edition, p. 30.

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Volume 14, Number 1 January, 1943

American Institute of Physics

Third Report of War Policy Committee*

December 7, 1942

Wartime Training in Physics

THIS is an attempt by the War Policy Committee to appraise the present situation as regards training in physics for war. We have endeavored to make the appraisal valid and informative, and hope that it and the suggestions arising from it may be useful to the War Department, the Navy Department, the War Manpower Commission, and other agencies now assuming responsibility for war training programs.

I. IMPORTANCE OF PHYSICS IN MODERN WAR

In this war, as compared with the last war, the combat forces are using a tremendous amount of complex physical equipment. Success in action now often depends on the reading of instruments and on the making of precise measurements. These are skills that are learned essentially in the physics laboratory or its equivalent.

What Physics Is Physics is the science of mechanics, properties of materials, electricity, magnetism, electronics, radio, optics, acoustics, heat, radiation of all kinds, and nuclear phenomena. The several branches of engineering are founded on one or more of these special fields of physics. A broad knowledge of physics is essential in engineering, medicine, and chemistry. A thorough knowledge of physics, where allied with personal qualities of resourcefulness, judgment, and the like, is perhaps the best possible equipment for solving technical problems which are new and unconventional in character.

War Role of Physics Every section of physics has an important role in modern warfare. Acoustics is employed against submarines; the threat of the magnetic mine needs no emphasis; new radio devices are used in numerous ways to find and destroy enemy aircraft and shipping; bombing and defense against bombing are very complex problems in mechanics, optics, and heat; the operation of aircraft and tanks depends upon acoustics and radio-communication; modern reconnaissance and photographic mapping present unprecedented optical problems. There are many other applications of great military importance.

* The War Policy Committee of the American Institute of Physics was established last spring to act and speak for the profession in matters concerned with the application of physics and the use of physicists in the war effort. Its two previous reports have dealt with the professional status of physicists, training in physics for the war, the use of physicists in the war, and manpower problems in physics. These reports may be found in the J. App. Phys. **13**, 345 (1942) and **13**, 471 and 477 (1942). The present report was prepared with the knowledge that drastic changes in the training programs of the War Department were in prospect but without much knowledge of the new plans.

Manpower Needs Physicists and engineers are needed to design and develop new and better weapons, instruments and methods of war, to assist in their manufacture or preparation, to use them in combat, and to instruct others in such use. It is essential to keep in mind these functions: research, development, production design, manufacturing, testing, operating, and training. Every one requires personnel trained in physics. Some of these functions require a very extensive training, some a moderate amount of training, and some only rudimentary training. The information is unfortunately not available to indicate just how many men are needed at each level of training in physics but it is quite apparent that it will be impossible with existing facilities to train enough to make up the deficit.

Survey of Needs Desirable A first step toward correction would be to get a quantitative appraisal of the situation. This would involve a systematic study of the trained manpower requirements of the technical tools which it is planned to use. Such a study would include, for example, a consideration of radars. A number of fully trained professional physicists are needed in the development of these devices; a large number of engineers and others with a good working knowledge of electrical physics and radio are needed for their production design and manufacture; a still greater number of men and women are needed for service and maintenance for which only a restricted study of some elements of physics is necessary; and a very large operating force is needed whose acquaintance with principles of physics and radio can be slight. The control of gunfire, the navigation of ships and aircraft, the sighting of bombs, the detection of submarines, and the forecasting of weather conditions may be mentioned also as requiring a study of the manpower needs at various levels of technical competence.

Present Deficit in Training Any investigation of the need for training in physics must take into account the fact that in recent years such training in the United States has been at a very low level as regards numbers and, in many schools and institutions, of low quality as well. We have entered a war making an abnormal demand for physics training with a deficit of such training and with teaching facilities not even adequate for peace-time. Our shortage adds insistence to the desirability of systematically analyzing our needs and establishing a training program of corresponding size and scope.

British Policy It is pertinent to mention that British experience in the present war has emphasized the importance of training in physics. This has led to the adoption of a policy in Great Britain which is highly significant even though it is probably not suitable for emulation in the United States. Under this policy the Ministry for War has, on its own initiative, asked the universities to continue and even to expand their instruction in physics, and the Ministry of Labour and National Service forbids the "calling up" of university physics students for any military service, other than flying, until the completion of their courses.

Distribution of Physicists It is also significant that here in America some 40 percent of the nation's personnel specially trained in physics is already engaged in war research and operations under the auspices of the Army, the Navy, and other government agencies. Another 30 percent is employed in industries of such a nature that they were among the first to be mobilized in the war production program. The remaining 30 percent consists of college and university teachers and this figure represents a considerable depletion. In none of these categories is the supply sufficient to satisfy the demand. This is especially true in the first field, war research. The National Roster estimates that a 50 percent increase in such personnel will be needed by the nation if present programs are carried out as planned. The percentages mentioned in this paragraph refer only to those who have *at least* one more year of training or experience than is represented by attainment of the bachelor's degree with major in physics.

II. STRATEGIC CONSIDERATIONS

Length of War The probable length of the war is the major strategic factor in determining the best policy for wartime physics training. A short war would call for quick mobilization of young men and the use of technical developments already initiated. Any new instrumentalities not yet started on the two or more year path from idea to wide application would be useless for a short war and no course of behind-the-lines training requiring more than one or two years to culminate could be considered. A long war calls for a very different plan in which "pre-physics" training and subsequent professional training are necessary to provide physicists to maintain the national capacity to develop new and superior weapons through the years faster than the enemy.

We still do not have available the facts necessary to predict the length of the war—if indeed anyone, even with all the facts, might do so! The difficulties in the way of any decisive United Nations offensive are enormous and there is, at this writing, no evidence that the final phases of the war are likely to come in any sudden or striking fashion. Although almost any effort would be justified if it gave assurance of an early victory, it is not obvious that the best effort we can make is good enough to gamble on. We must not at the moment expend *all* possibility of having the results of advanced physics training three or four years from now. Something must be reserved. On the other hand, due allowance should be made for the possibility that an active soldier in combat with fair weapons immediately may accomplish more for victory than two soldiers with better weapons two years from now. In this war "too late" has perhaps been worse than "too little."

Time Available for Training Another strategic factor, not unrelated to the first, is transportation. According to a statement made by General Marshall about November 1st, some 800,000 men were transported overseas in the ten months following Pearl Harbor. Doubtless the rate of transport can be increased rapidly but this is a factor which limits the speed with which men can be thrown into action, whatever assumption is made as to the length of the war. The period available for training some of them before they sail will necessarily be long. The very difficulty of sending men emphasizes the importance of giving them the best possible training and equipment.

Importance of Superior Weapons Another strategic fact is that on certain fronts there are already large United Nations forces and the principal need on these fronts is for more and better equipment such as we in America can devise and produce. It may be at least as important to keep the weapons on these fronts superior to the enemy's as it is to open new fronts. It must not be forgotten that we are the "arsenal of the democracies."

On all fronts in modern war, technical weapons and methods are a factor of increased effectiveness when compared with numbers of men. England's R.A.F. with their Spitfires and their radiolocators repelled an invasion. It is a question whether, without them, the entire British Army and Navy, could have repelled it. Germany's submarine warfare, with relatively few men and no vast equipment, effectively put out of action, or diverted from other action, all of the ship crews lost, the shipbuilders needed to replace the ships sunk, the allied troops desperately needing the equipment that was sunk or delayed, and the crews of convoying war ships.

Without predicting that physicists and engineers will produce miracles or easy victories, we can say that new and superior weapons already devised by them and now being produced will give our forces next year markedly greater combat strength than numerically equal forces possess now. It is the physicists and engineers who can give our soldiers and sailors an advantage over the soldiers and sailors they face, and enable them to achieve victory quicker and with fewer casualties. It is probable that superior training and equipment will do more to maintain the numbers of our men in action by preventing casualties, than will the cure and restoration of casualties after they happen. In this sense the services of the nation's physicists may well be much greater than the services of the nation's more numerous medical men.

III. TYPES OF TRAINING NEEDED

Bearing in mind the nature of modern total war, we can see clearly the need for men and women with all degrees of training in physics. The following classification is useful:

Training for Routine Operations *1. Men trained in the elements of special subfields of physics.* Such men trained in the elements of electricity or mechanics are wanted in large numbers by the Air Corps, the Signal Corps, and the Navy. Such training can be given to high school seniors or in short intensive schools in Army camps or at colleges under contract to furnish facilities. The shortage of really experienced teachers is such that reliance must be placed on less experienced teachers using simple manuals of instruction, and on the use of the better students of each class to help teach the next. There are not enough specialists in physics to spare many for such teaching.

Pre-Training for Technical Services *2. Men who have had quick survey courses in general physics.* Such courses can be given in high schools or colleges. They require trained teachers and cannot be given well by teachers who have not previously taught physics. Such courses, together with mathematics, are excellent preparation for the Air Corps, Signal Corps, Artillery, Navy, weather forecasting, and other military duties. It is a national calamity that the teaching strength of our high schools is so inadequate to meet the present urgent need for elementary training in physics for a large number of men and women.

Pre-Training for Technical Services Continued *3. Men who have had the general college course with laboratory.* Such courses require the best teaching services available. They are so much better than the survey courses without laboratory that the latter type should not be given with college grade teachers and facilities. Men who have taken these courses will operate, maintain, and even supervise the operation of radars, bombsights, fire control equipment, and other technical devices too numerous to mention. With a little specialized training such men will be extremely useful officers.

Training for Technical Services *4.* In general some special training will be called for in addition to type 3. The general course is the best prerequisite for weather forecasting, air navigation, operation of short-wave radio devices, and the location of submarines. Each of these and other applications calls for good stiff special training for which physicists, radio-engineers, meteorologists, and others of full professional competence are required as teachers.

Physics Training *5. Pre-physics students.* Only after completion of the general college course may qualified students begin professional training. This is the equivalent of the beginning of pre-medical training and is likewise the stage when students first really begin professional training in other sciences. A curriculum majoring heavily in the chosen subject is followed. For physics it is recommended that the requirements for a major consist of a minimum of 22 semester hours of subject matter work in physics of which at least 8 semester hours must be in courses each of which requires calculus as a necessary prerequisite and makes use of calculus throughout the course in the presentation of the subject matter of the course. Such a curriculum can be condensed to its essentials and accelerated considerably. Graduates are acceptable helpers in government war research agencies and industrial research laboratories. They are in great demand by the war industries. They are "up to" nearly all kinds of technical military operations. They can be used, sometimes even before graduation, as teaching assistants in general physics courses. From this group physicists are recruited for further training or experience leading to full professional competence. This is a highly selected group and only about 1200 are graduated annually by American colleges and universities.

Condensed Emergency Training *6.* For apt students it should be possible to combine the general physics course and a selection of the most essential advanced courses into a concentrated curriculum which would be completed about two years after graduation from high school. Men and women who completed

such a curriculum would be very valuable in all kinds of experimental work in industry, war laboratories, and military operations. They would not be qualified to become professional physicists without further broadening and maturing.

Full Professional Training *7. Professional physicists.* The accepted standard of training for the profession of physics is the Ph.D. degree or its equivalent, requiring several years of professional training or experience of a high order after attainment of the bachelor's degree. Not more than 400 individuals per year normally attain such standing. Professional study is commonly combined with part-time teaching and research experience. At present research experience may be obtained on many war projects. Only the larger institutions are well equipped to offer professional training. Its supervision requires a high degree of scholarship on the part of the faculty and extensive laboratory equipment must be available.

IV. AVAILABILITY OF TEACHERS

For High Schools With few exceptions, teachers of high school physics do not qualify as physicists. They have not carried their subject matter training far enough. Nevertheless the teaching of physics at the high school level is an exacting task requiring a high standard of competence. There are not nearly enough suitably trained teachers available and many of these are being attracted into highly paid positions in war plants. It is not easy for the teacher of other subjects to become a teacher of physics. Any national plan for expansion of high school physics training is thus confronted with serious limitations.

For Colleges Teachers of physics at college level are in demand for war research work and for Signal Corps, Air Corps, and other special Army and Navy services. So great a fraction of the teaching strength of colleges and universities has already been diverted that it is now necessary to economize on the need for teaching or else give serious consideration to a transfer of physicists in war research and war operations back to teaching. The "conversion" of teachers from other fields is feasible only with apt candidates and most of these are in fields in which there is also a shortage of personnel.

V. AVAILABILITY OF STUDENTS

High Schools In the senior high school year there is no reason to doubt the availability of all the students that can be taught. Widely publicized campaigns of the U. S. Office of Education, the Navy, the Army Air Corps, and other agencies all place primary emphasis on the study of physics. State Superintendents of Education are furthering their campaigns. Even the American Legion, through its local posts, is urging the expansion of high school training in physics.

Colleges The availability of students to receive training at college levels is controlled primarily by the operation of the 18-year draft law. Many students are ready for college at 17 but the bulk are not. It follows that students for the general college physics course (open to first year men) will be greatly limited unless (a) a workable system of draft deferment is put into practice, or (b) men already inducted are detailed to colleges and universities for instruction. If some kind of student training corps were to be established it would amount to one or the other of these alternatives or a composite of both. In any case some kind of selective machinery is essential and it would have to select on the basis of aptitude tests and scholastic records often without presuming any previous study of physics.

Deferment or Detail At present the Selective Service System does not provide for deferment of physicists-in-training until they have entered upon more advanced training than the general first college course. At this writing no information is available concerning any plans of the War and Navy Departments beyond the Enlisted Reserve Corps now operating. The War Department (but not the Navy Department) has announced that students in the Reserve will in general be called to active duty "at Selective Service age." It is, of course, pos-

sible that students in certain important fields including physics would not actually be called out. It has also been suggested that at or after induction into the Army, a number of men might be selected for "assignment" to college.

Basic Conditions to be Met by any Plan *For maximum value in the war, any plan providing student status for young men should provide for their selection solely on the basis of their ability, should avoid dependence for its operation upon their financial resources, should eliminate any basis for the "slacker" charge, and should attempt to reduce any restlessness for "more active" participation in the war.*

"Non-Military" Students It must be remembered that in total war the training of women and of men not physically acceptable for military service is as important as the training of men the Army and Navy could use directly. It is not desirable to make distinctions between the training programs where it can be avoided. It must also be remembered that civilian war agencies and industries cannot, in such a field as physics, get along on recruiting only from women and physically unfit males. These represent too small a percentage. Normally only about 5 percent of physics majors are women. The percentage of physically unfit is smaller for college men (and hence for physicists) than for the male population as a whole.

VI. CONCLUSIONS AND RECOMMENDATIONS

1. General While ample evidence is available to show in general the importance of training in physics for the present war and the need for a great expansion of such training, there is insufficient evidence of the specific nature of the need as related to actual war uses for trained personnel. Studies by the War Department, the Navy Department, and the War Manpower Commission should be made to determine the types of training which should be provided and the numbers of individuals needed with each type and degree of training.

Admitting the urgent need to get quickly into action the largest possible number of men from 18 to 22 years of age, it is our opinion that it would be a national disaster if all training in physics at the college level were stopped. A national mechanism should be developed for the supervision of such training. If it is at all practicable, there should be one mechanism for all important technical fields under one joint sponsorship (not unrelated programs under the War Department, the Navy Department, and the War Manpower Commission). Any such mechanism should provide for the selection of students for special training on the sole basis of their ability to become more useful to the nation through such training and should be independent of ability to pay college tuition and subsistence. Students should be assured an honorable status in the war service of the nation.

Students selected for advanced training in physics should concentrate their efforts on physics courses with no additions except the necessary mathematics prerequisites and a course in general chemistry. Laboratory practice should be emphasized throughout. Upon completion of each stage in training, a reduced number should be selected for further study. Even under present conditions, some students should be carried to the level of professional competence. During the latter part of their professional training such students could help with teaching or war research. Conversely we recommend that teachers and war research workers continue, where possible, their professional training.

It will probably be necessary, in order to insure the fullest use of the capacities of qualified teachers and of laboratory equipment, to transfer teachers and equipment from some institutions to others.

It is difficult to find additional steps to help the high school situation beyond deferment of full-time teachers of physics as provided in Selective Service Occupational Bulletin No. 23 and provision of E.S.M.W.T. courses for teacher training as already authorized. The problem should be accorded further study.

2. Specific As a first approximation to a definite working program, admittedly open to adjustments, modifications, and the substitution of alternate

features, we propose the following:

(a) Continuation of present policies and provisions for stimulation of high school training in physics.

(b) Continuation of present special schools of the Army and the Navy in which enlisted and commissioned personnel are taught the operation and maintenance of technical equipment with only the elements of the underlying scientific principles. We are convinced that this training can be carried on by engineers, the better graduates of the schools themselves, and a small complement of physics teachers—great reliance being placed on detailed training manuals. Probably not more than 300 or 400 physics teachers will be needed for all such schools and some of these have been recruited already.

(c) Establishment, at suitable colleges and universities, of an accelerated college grade program with students ultimately supplied by the Army and the Navy in an enlisted status, contractual arrangements being made to provide subsistence and training. Please refer to appended "Notes on Training at the College Level."

NOTES ON TRAINING AT THE COLLEGE LEVEL

Any plan for wartime training in physics at the college level must depend on the needs of the armed services and can only be worked out in conference with representatives of the latter. Hence the Committee hesitates to put forward any specific curriculum although it has actually drawn up several alternate proposals and is prepared on short notice to assist in detailed planning of curricula and course contents.

Presumably a general course in physics with laboratory would be required for a large number of students. The Committee feels that physics courses, to have value in the war, should be strong subject matter courses with laboratory. Assistance to institutions in obtaining priorities for the purchase of teaching supplies and equipment will be necessary.

Presumably also a suitable number of students would be selected for further training in physics as a specialty. It is possible to plan a program of such special training for almost any term length and any number of terms which would in each stage greatly increase the war value of the students.

At all levels, including the first general course, it is proposed that all physics students at any institution be required to take identical courses whether they are members of the Army or the Navy Enlisted Reserve, civilians, or students in any other status. Women students would also take these same courses. Moreover it is not recommended that students ineligible for military service be selected for further training on a more lenient basis than those who are. The time of teachers and the facilities of physics laboratories must be conserved for the ablest. It must be emphasized that these same teachers and laboratories can often be employed for the creation of new and superior weapons of combat. They are a very rare "commodity" and must be rationed out where they will do the most good.

It would be important to consider carefully the assignment of those students who successfully complete the highest levels of the collegiate program. Some should go into the armed services but this group would be the only source of recruitment of teachers and of men suitably trained for war research agencies such as the Naval Research and Ordnance Laboratories, the National Bureau of Standards, the Office of Scientific Research and Development, and others. A large percentage of these students (probably a majority) should be discharged or detailed to inactive status in such agencies, or assigned to teaching work or further study. Such further study would have to be provided in a few institutions. It would comprise courses in theoretical and experimental modern physics of the type normally available to college seniors and graduate students.

It is recognized that after the war a rounding and maturing course would be needed by any participants of this program who wished to follow physics as a career.

Submitted by the War Policy Committee:

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Electrostatic Electron Microscopy. I

By C. H. Bachman and Simon Ramo

Electronics Laboratory, General Electric Company, Schenectady, New York

This paper, consisting of three parts, describes investigations made with the objective of developing a simplified, practical microscope of the type which yields magnified images of transparent specimens with a resolving power superior to that of the best light microscopes. The first part deals with the general problem of design, including the electron gun and the imaging lenses. A later part will describe a completed instrument embodying many of the results of the investigations.

IT is the purpose of this paper to describe investigations made by the authors toward the development of a practical, commercial electron microscope of the type which yields magnified images of transparent specimens with a resolving power superior to that of the best light microscopes. Also, a completed instrument will be described which embodies many of the ideas and principles that were the subject of investigation. The first portion of the paper will not be concerned with any one instrument design but rather will be a general discussion of the problem of design of electrostatic electron microscopes. Although the authors' investigations dealt mainly with the electrostatic method, the magnetic lens type of microscope will occasionally be referred to for comparison purposes.

Many papers have been published both on this continent and abroad on electron microscopy. These papers have been mainly of two kinds: Either a completed instrument has been described, the discussion centering on the instrument's chief performance characteristics and its physical form; or more or less general theoretical discussions have been presented of electron optical systems. Little has been published about what might be called the "engineering" of an electron microscope.[1] While the following is far from a complete dissertation on the optimum design of an electron microscope, an attempt has been made to consider the basic problems which are met in the building of a commercially practical instrument. These problems will be described and a partial solution to many will be given. It is thus hoped that the article will point out the various difficulties of theoretical, experimental, and engineering design nature which arise in microscope development and what might be done about those problems in certain cases.

ELECTROSTATIC *VERSUS* MAGNETIC LENSES

It is known that both electric[2] and magnetic fields[3] can be used to obtain electronic images of high resolution. The authors considered both general types of microscopes as to their suitabilities for a commercial instrument of wide range applicability and also as to the potentialities of each in advancing closer to the ultimate resolving power. The investigations and the sample instrument described in this paper bear out the belief that the electrostatic type is admirably suited to the role of the practical commercial instrument, possibly even more so than the magnetic type because of the inherent simplicity of the former. It is possible by using electrostatic lenses to remove at once one of the most complex components of the electron microscope, the closely regulated power supply. The focal length of a magnetic lens is a function of both the magnetic field strength and the velocity of the electrons being focused by the lens. The magnetic field is derived from a current and the electron velocity from a voltage so both must be very closely regulated or else the focal length will vary too greatly and the image will be blurred. Regulation tolerances of the order of one part in many thousand have been found necessary.[4] In the electrostatic system, the electron velocity and the lens focusing action may be derived from a single high voltage source. The lens focal length in this case is then mainly a function of its physical size and configuration since an increase or a decrease in electron velocity is always accompanied by an

increase or decrease in the electric field's focusing action of just such strength as to yield precisely the same electron paths, except for a small relativity effect.

As the lens components in the electrostatic system are of relatively small size and simple configuration, it becomes practical to use multiple lenses held together in a compact enclosure which greatly decreases the volume and area of the chamber to be evacuated.

The authors, however, do not take the view that the investigations reported here constitute a proof that one or another microscope is superior. They feel rather that the field is young and important advances may be expected in each type of instrument, so that investigations of all microscope attacks should be continued and an open-minded attitude should be used in making comparisons between these various attacks.

SCHEMATIC DIAGRAM OF ELECTROSTATIC MICROSCOPE

Figure 1 shows the major electron-optical components of an electrostatic microscope, each of which will be discussed in the remainder of the paper. All parts are at ground potential except the cathode, its shield, and the central electrode of each lens. Two stages of magnification are shown but, as will be explained later, it may prove advantageous to use three or more stages.

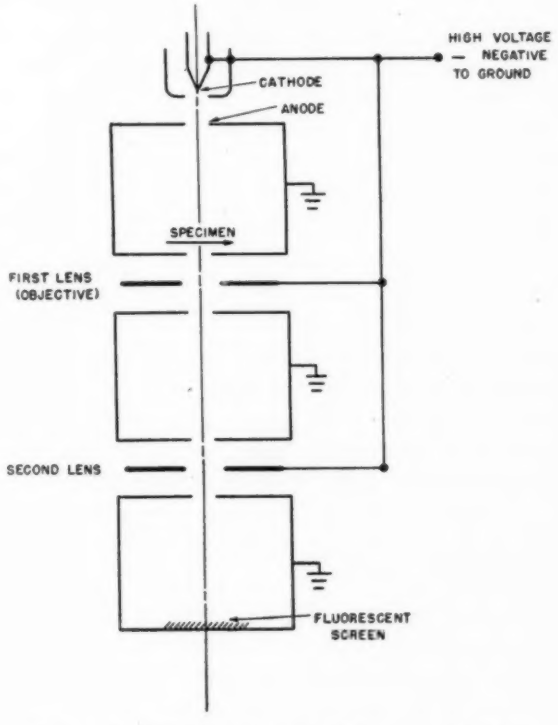

FIG. 1. Diagrammatic sketch of components of an electrostatic microscope.

THE GUN PROBLEM

It is convenient to divide the electron optics of the microscope into two parts: (1) The electron gun, which serves to illuminate the specimen; and (2) the imaging system which magnifies the electron image of the specimen into a larger electron image on the fluorescent screen or the photographic plate. For a simple commercial instrument, it is especially desirable that the gun should be reduced to as few components as possible, consistent with obtaining the resolving power and other performance capabilities desired for the over-all instrument. Since it is possible to operate the imaging system of an electrostatic microscope with only one voltage difference to ground, it is also preferable in the interest of simplicity to avoid those electron guns which require additional voltage sources. Although the authors performed some experiments with multivoltage guns, a single voltage electron gun was developed which proved quite satisfactory in all respects and this type will be discussed here.

The first basic requirement of the electron gun, regardless of whether the imaging system is magnetic or electrostatic, is that it be capable of producing sufficient current density at the specimen. The cross-sectional area of the electron beam at the specimen and the angle which the electrons make with the axis must also be considered carefully. An important consideration in this regard is the heating of the specimen by the electron beam. It is desirable to obtain as bright a final image as possible for visual observation with a minimum of current striking the specimen. This means that it is disadvantageous to illuminate the specimen with electrons if those electrons, for one reason or another, cannot possibly contribute to the contrast or even appear in the final image at any time. It would be pointless, for example, to irradiate a large part of the specimen if only a small portion on the axis is to be imaged. Also (for the conventional bright field type of illumination) there would appear to be

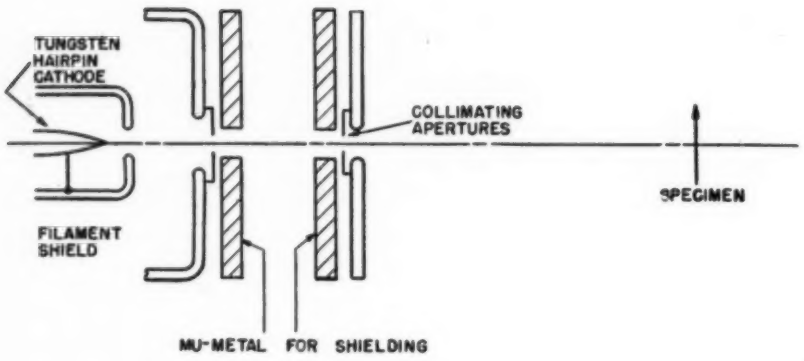

FIG. 2. Arrangement of electrodes in unipotential electron gun.

no reason to aim electrons at the specimen if their incident angle is such that in the absence of deflection they will surely be rejected by the objective lens aperture stop. Of course, the contrast obtained in the final image is the result of the distribution of the electrons arriving there; but every electron should be granted equal opportunity to contribute to the final image of that part of the specimen through which it passes, unless the specimen itself decides otherwise and absorbs it, slows it, or deflects it. Any electron that impinges upon the specimen with so large an angle that in the absence of preferential deflection by the specimen it is certain to be collected at a lens aperture only serves to heat the specimen uselessly.

There is some evidence to indicate that for very thin specimens, the bulk of electrons that come through the specimen do so with their slope virtually unchanged. Thus there is a possibility that the limiting angle of the electron bundle leaving the objective lens can be set mainly by the gun apertures rather than by apertures of the objective. Sometimes this type of limiting is an easier task mechanically because the last aperture in the gun preceding the specimen determines that angle, and that last aperture may be more favorably located than the objective lens stop so as to avoid the need for producing extremely small diameter holes. It is probable that the lens aberrations as well as the particular thickness and material of the specimen that is under examination have an important part in determining whether the gun angle can serve as the effective stop of the system or whether that limit is best determined by the first imaging or objective lens. However, our best results have been obtained with a small gun angle and, although the information along these lines cannot yet be considered complete, indications are that the gun angle has a more easily discernible effect upon the resolving power than does the lens aperture.

The separate condenser lens for focusing electrons on the specimen is often considered a necessary component of an electron microscope. Any gun, since it is a means for doing such focusing, must include in it the focusing properties of a condenser lens. However, it is possible to use an additional condenser lens and electrons coming from a simple gun which might in itself be fairly successful in illuminating the specimen can be additionally focused to give even a more intense concentration on the specimen. Such a separate condenser lens has been used by the authors, but the method of visual observation and photography of the image finally chosen for the instrument to be described later was such that it did not seem necessary to add this additional lens to increase the intensity. Once adequate intensity in the image is obtained additional electron beam current simply serves to damage the specimen and add the possibilities of undesirable space-charge effects in the objective lens.

If, because greater intensity at the specimen is desired, a condenser lens is deemed necessary, it is very easy to add it to the gun design. With proper collimation of the electrons coming from the simple electron gun, a parallel beam can be considered to enter the condenser lens. The rays leaving this lens will focus at a desired point on the axis if the condenser lens is simply located at a distance equal to its focal length ahead of that point.

The simple design of gun indicated diagram-

matically in Fig. 2 was chosen by the authors as coming closest to satisfying the general requirements. The proper relation between filament point location, filament-shield aperture diameter, anode-filament spacing, and the spacing and size of the other apertures depends upon a number of factors that will be peculiar to each instrument design, such as the exact location of the imaging lens system and the specimen, the filament life desired (filament life being balanced against intensity of illumination), the accelerating voltage, and the extent of magnetic shielding of the filament current's magnetic field which it is practical to use. In general, it is necessary to adjust the filament location and filament-shield aperture diameter experimentally for each different construction. The information now available on axially symmetric immersion lenses, such as are used in common cathode-ray tubes, is not applicable to the case of the asymmetrical hairpin-type cathode used in the microscope.

An important problem in the electron gun design is overcoming the magnetic field due to filament current, whether it be supplied by d.c. or a.c. The question of magnetic shielding of the instrument will be discussed in more detail later but for the moment it is pointed out that although cross magnetic fields (those components perpendicular to the axis of the instrument) can be detected by the resolution variation with azimuthal angle about the axis, the axial field causes a bad effect not so easily traced to the filament current.

CONSIDERATIONS AFFECTING CHOICE OF MAGNIFYING LENSES

The electron lenses which form the imaging system of the electron microscope must possess

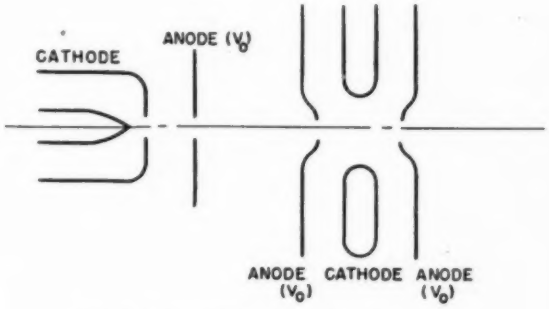

FIG. 3. General form of unipotential lens.

the following general characteristics:

(1) The focal length should be positive and short without resorting to minute dimensions.

(2) If the lens is to be the objective lens, then the "in-focus" position of the specimen must be external to the high field region of the lens. As a matter of fact, it will generally be absolutely necessary to insure that the specimen is in a region which is, practically speaking, entirely free from electric fields.

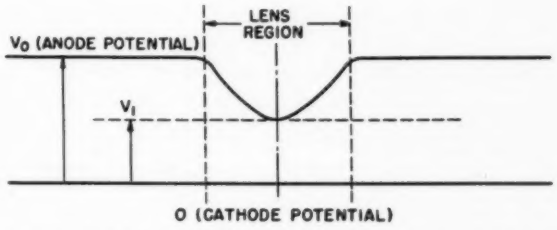

FIG. 4. Axial potential distribution of unipotential lens.

(3) The electrodes should have such physical form that the insulation problem will be minimized and field emission at the operating voltage cannot take place.

(4) The lens should preferably be symmetrical in general physical form with a single potential to ground, namely, that of the cathode, sufficing to energize the lens.

(5) There should not be an excessive drop in axial voltage in the central section of the lens because, as will be shown later, both the stray magnetic field and chromatic aberration effects increase with this drop and they may easily become excessive.

(6) The lens should be readily manufacturable to the required precision.

(7) Spherical aberrations and other purely geometric defects should be low enough to yield the desired resolving power without necessitating the use of impractically small lens stops.

It is quite evident that with all of these requirements to consider, it is not a simple matter to arrive at the optimum lens design. However, certain of these seven factors may be taken as basic or essential and others may be regarded as ideals toward which one should work. For example, the requirement that the lens electrodes must be either at ground (anode) or cathode potential may be taken as a fundamental point. Then, the schematic diagram of Fig. 3 shows

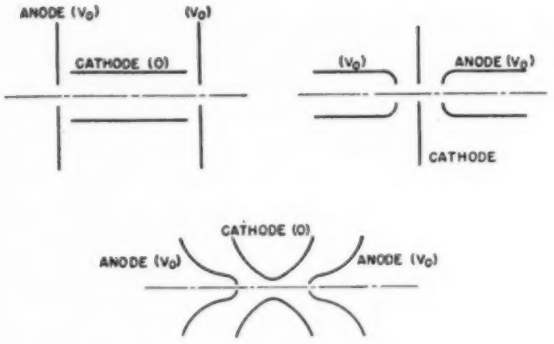

FIG. 5. Unipotential lens configurations.

the general form which the lens might have. Shown in this figure are three axially symmetric apertured conductors, the two similar electrodes being at ground or anode potential and the central electrode being at cathode potential. With such an arrangement the potential distribution along the axis will take the general form indicated by the diagram of Fig. 4. Here the anode-cathode voltage is called V_0 and the minimum voltage at the lens center V_1. The use of small axial holes in the grounded electrodes will insure that the electric field will die off rapidly with distance outside the lens region where, in the case of the objective, the specimen will have to be placed. These holes may also be the lens "stops." What is next needed is to determine more precisely the shapes and spacings of these electrodes and the way in which they are to be mounted so that the requirements stated above will be satisfied. Plane and cylinder combinations or more complex curves (Fig. 5) may be used. The problem is one of synthesis, and hit-or-miss experimentation cannot be relied upon exclusively to yield optimum results.

Now, the focal properties of a lens are completely determined when the distribution of potential along the axis is specified. Figure 4 gives the general form of potential distribution which the lens must have, but the particular distributions that are possible are infinite. It is accordingly desirable to know what is to be the preferred distribution by computing focal properties for a variety of theoretical distributions. A convenient way to attack this problem is to choose a simple mathematical expression for the axial potential inside the lens region that contains parameters which, if varied, will yield a variety of lens voltage distributions. In this way it would be expected that certain basic relations would appear that would serve as guides to the choice of the most practical lens design.

Now, there are several simple ways in which the potential can vary, some of the simplest being: (1) The axial potential can increase linearly with axial distance from the lens center until it reaches anode potential V_0, at the entrance and exit of the lens (Fig. 6). This distribution is closely approached in a physically realizable lens consisting of three plates at potentials V_0, V_1, and V_0 equally spaced and possessing very small axial holes. (2) The axial potential may vary sinusoidally with distance according to the function

$$V \propto 1 - c \cos \omega z, \qquad (1)$$

where z is the axial distance from the lens center. By varying the two parameters, c and ω, a large variety of interior lens distributions are obtained (Fig. 7). (3) The axial potential may vary exponentially with axial distance from the lens center. Since we are restricting the discussion to symmetrical lenses, the simplest exponential type of variation probably is:

$$V \propto A\left[\frac{\epsilon^{\tau z}+\epsilon^{-\tau z}}{2}\right] = A \cosh \tau z. \qquad (2)$$

The two parameters A and τ provide further

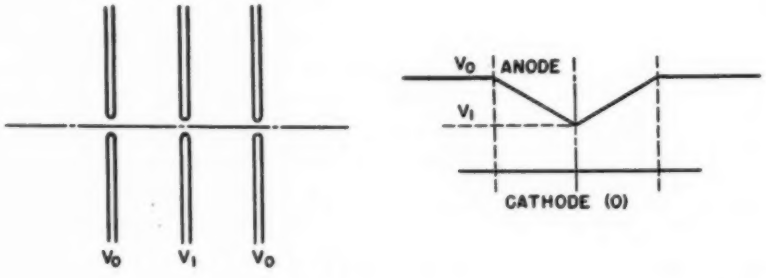

FIG. 6. Lens with nearly uniform fields between electrodes.

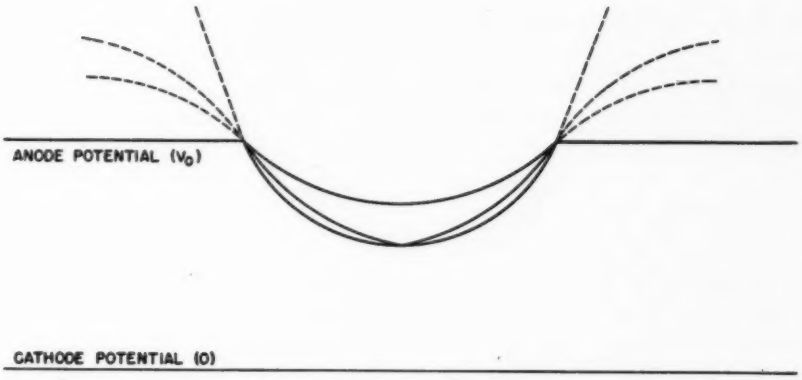

FIG. 7. Axial potential distributions in the cosine-Bessel lens.

wide varieties in distributions. The two general lenses corresponding to Eqs. (1) and (2) are physically realizable. From the general theory of electrostatics,[5] it is found that there are solutions to Laplace's equation, which all field distributions must satisfy, that correspond to the axial distributions of Eqs. (1) and (2). These are:

$$\Phi \propto 1 - cJ_0(i\omega r)\cos \omega z \qquad (3)$$

and

$$\Phi \propto A \cosh \tau z J_0(\tau r), \qquad (4)$$

where J_0 is the Bessel function of zero order and r is radial distance from the axis. With the equations for potential known, one has only to equate Φ to V_0 to determine the equation for the surface of the two anode-voltage electrodes, and to equate Φ to zero to obtain the form of the cathode potential electrode. Two examples are shown in Fig. 8.

METHODS FOR COMPUTING FIRST-ORDER FOCAL PROPERTIES OF LENSES

There are a number of special methods which apply particularly well to the type of lens of interest in an electrostatic microscope. For one thing, the boundary apertures will always be relatively small in diameter as compared to their focal length so that the aperture lenses thus formed may be treated as thin lenses superimposed on the entrance and exit of the lens proper. These two lens actions at the apertures are divergent;[6] their focal lengths are easily computed[6] after an estimate is made of the field existing inside the lens near the aperture. (The field on the other side is taken as zero.) The center of the lens is a relatively highly convergent system. This central lens region can be treated by thick lens formulas[6] or as a combination in itself of three thin lenses. In the case of the "uniform field" potential distribution of Fig. 6,

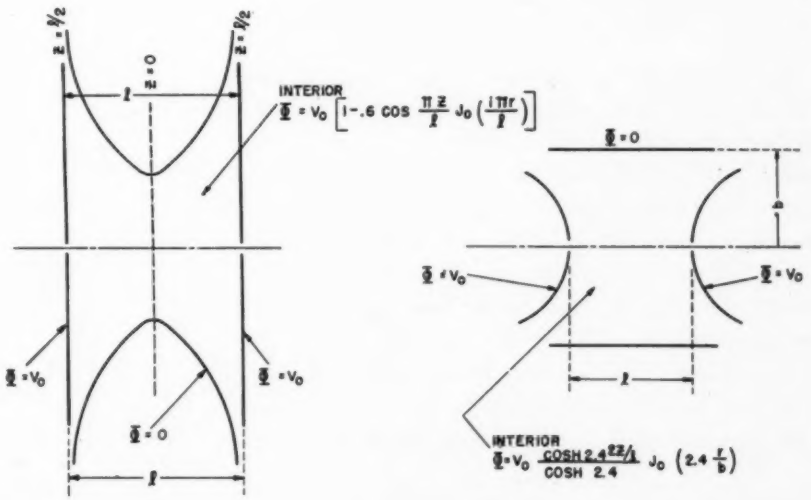

FIG. 8. Examples of cosine-Bessel and cosh-Bessel lens electrode configurations.

the central of these three lenses is regarded as simply another aperture lens, separated from the entrance and exit apertures by uniform fields whose effects are easily computed.[6]

Because of the symmetry of the problem, it is often easier to trace two particular rays from the center of the lens to the exit, rather than to start the analysis from the entrance and carry it through to the exit. One of these two specially chosen rays (Fig. 9) passes the center of the lens region with a slope toward the axis of zero at an arbitrary distance from the axis; the other has an arbitrary slope at the lens center and it crosses the axis there. The complete path of the first-mentioned "parallel" ray theoretically must, from symmetry, intersect the axis on both the entrance and exit of the lens at the same distance to the lens center. From simple geometrical optics, the distance from either of these intersection points to the corresponding principal plane must equal twice the focal length of the lens. If the principal plane location were known, then the focal length would follow from the location of these intersection points. The second (central) ray, from symmetry, must enter and leave the lens region making the same angle with the axis. If its path on leaving or entering is extended to intersect the axis, this intersection point is a principal point of the lens. Thus from these two "half-paths," the focal length and principal point location of the lens are obtained. We do not then have to make computations of the complete lens region, but only half of it.

FIRST-ORDER PROPERTIES OF PRACTICAL LENSES

By varying the parameters of the three potential distribution functions which have been described, the uniform field, cosine, and cosh, data which are applicable to a practical electron microscope were obtained. For the objective lens, the object position must fall outside the lens so lenses whose focal points fall at a distance less than $l/2$ from the lens center are eligible only for the second and later lenses. The chief criterion of a lens for focal lengths longer than $l/2$ was found to be the voltage drop, V_0-V_1, in the lens. For the same voltage drop, up to about

$$(V_0-V_1)/V_0=0.5,$$

all variations of the three lens distributions considered yielded very close to the same focal lengths. Furthermore, the principal plane locations were found to be very close to the center of the lens for all three cases. This latter was especially true of the "uniform field" lens and the cosine-Bessel lens with the principal planes of the cosh-Bessel lens lying within about 5 percent of the lens length from the lens center. The critical focal length ($f=l/2$) occurred for a voltage drop of between 40 and 50 percent of the cathode-anode voltage.

Accordingly, as far as the objective lens is concerned, there seems little reason to the authors to seek out special voltage distributions which might offer shorter focal lengths while still allowing the object to be placed outside the lens region. For lenses other than the objective lens, the study must be regarded as much farther from complete, for with the lens voltage drop exceeding around 50 percent of the cathode-anode voltage, the principal planes move away

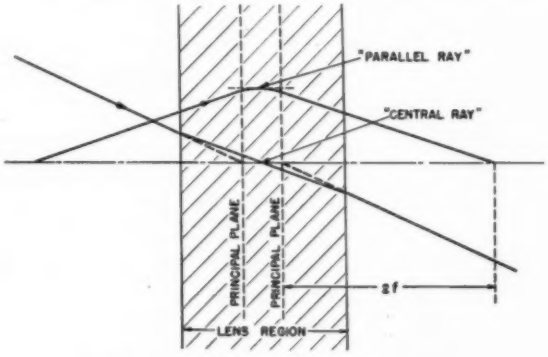

FIG. 9. Rays in a symmetrical lens.

from the lens center and it becomes less accurate to say that the focal length depends almost entirely upon the voltage drop. It is quite possible that lenses other than the objective might be built with exceedingly smaller focal length to lens length ratios than have been used thus far by the authors. However, there are manufacturing simplifications in building all lenses the same. Also, as has already been mentioned, the chromatic aberration of an electrostatic lens and the effect of stray magnetic fields may become objectionably large if large voltage drops are used, so that both of these factors constitute further limits to the decreasing of focal lengths.

With the knowledge that varying lens poten-

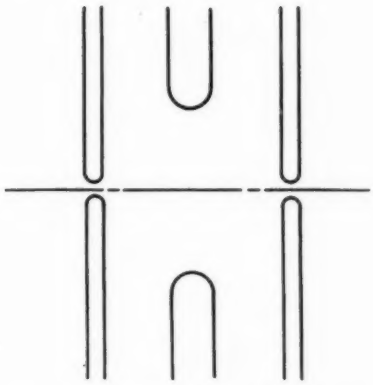

FIG. 10. Practical lens electrode configurations.

tial distributions, unless they depart quite substantially from that pictured in Fig. 6, will result in no huge improvement in the ratio of focal length to over-all lens dimensions for the objective lens, one is then left with the second-order properties to consider in choosing a lens configuration as well as the very practical problems having to do with insulation and possible field emission from the electrode surfaces. From the latter standpoint, further study and experiment indicated that field emission need not occur at voltages of 25 kv to 50 kv so long as no attempt is made to shrink dimensions below those yielding a focal length of about one centimeter. The most serious practical problem is that of insulating the central electrode at its support. The insulator must, of course, be capable of being machined to hold the lens in the required precision of alignment and the shape and type of material must satisfy vacuum requirements. Both of these factors may in turn affect the choice of the lens electrode configurations. Consideration of these factors and a number of experiments finally resulted in the choice of lens having the general configuration shown in Fig. 10. Figure 11 shows in a general way how the focal length varies with the chief lens dimensions.

BAND SPREAD OF FOCAL LENGTHS DUE TO ABERRATIONS

We shall consider next the matter of certain second-order effects or aberrations. Electron-lens aberrations, particularly chromatic and spherical, have been studied often,[8] but the discussion of the problem (particularly as applied to electron microscope design) is not likely to be complete for some time. What is needed is an expression for resolving power as limited by these aberrations in terms of the design parameters of the microscope-lens system: (1) lens configuration, regardless of its physical size, i.e., lens type; (2) focal length, which for a given geometry fixes the lens dimensions; (3) number of stages for a given magnification, or object and image distances; (4) lens-stop radius. It is conceivable, for example, that lens number 1 may have inherently less aberration than lens number 2 judged by keeping all design parameters constant (except, of course, the form of the electrodes). But lens number 2, simply by virtue of its different geometry, may make practical a superior choice of the other parameters (as, for example, physical dimensions for a given sparking voltage) and the final lens design for a particular microscope application will finally prove the number 2 lens to be the one with the higher resolving power.

If an expression for the band spread of focal lengths for electrons ranging from those extremely close to the axis to those limited by the lens opening is derived, then a convenient expression for the resolving power in terms of the microscope design parameters can be easily set up. Figure 12 shows electrons leaving the object at P and arriving at the image which is spread between Q_1 and Q_2, a distance Δq apart, due to the spread in focal length, Δf, between electrons near and far from the axis. If p and q are object and image distances to the proper principal planes, as shown, then elementary

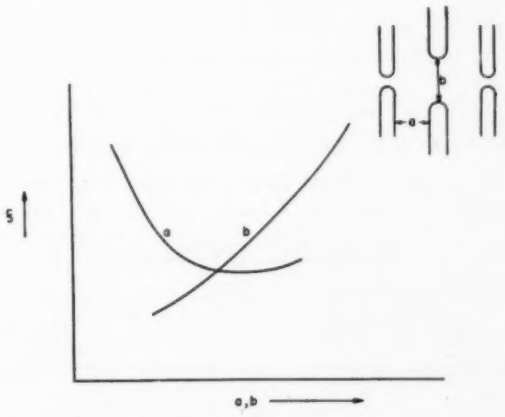

FIG. 11. Dependence of focal length on lens dimensions.

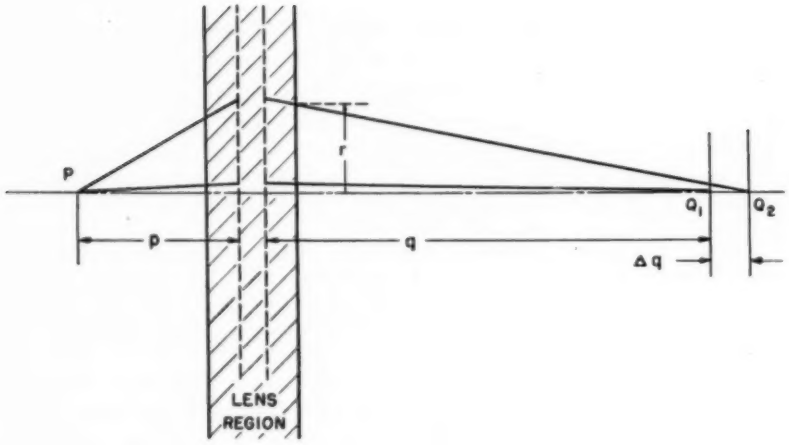

FIG. 12. Variation of image distance with separation from axis at lens exit.

optics states that

$$1/p+1/q=1/f,$$
$$1/p+1/(q+\Delta q)=1/(f+\Delta f).$$

Subtracting the two equations and simplifying gives the approximate result

$$\Delta q=q^2/f^2\Delta f.$$

From this longitudinal aberration the radius of the circle of aberration in an image plane placed halfway between the extreme positions, Q_1 and Q_2, will be about

$$r_q=(q/2)(\Delta f/f^2)r.$$

There may be some question as to just where between Q_1 and Q_2 the image plane should be placed. The optimum location will vary in general with the distribution of electrons in angle and the type of aberration. The above equation should give correct order of magnitude in any case.

Dividing by the magnification we obtain an approximate expression for the resolution (diffuse circle radius in the object plane)

$$\delta=(p/2)(\Delta f/f^2)r. \tag{5}$$

We are now ready to consider the magnitude of Δf for various lenses and types of lens aberrations.

SPHERICAL ABERRATION

In the appendix to this paper, it is shown that the spread in focal lengths due to spherical aberration may be expressed in the following form:

$$\Delta f=Sr^2/f, \tag{6}$$

where r^2 is the radius of the stop at the lens exit, f is the lens focal length, and S is a dimensionless constant characteristic of the lens electrode configuration. For a given lens "type," S is fixed—then choice of any dimension determines f or choice of f determines the dimensions. Step-by-step computations for S, as sketched in Appendix A, were made for two lens potential distributions, one a cosine-Bessel lens and the other a cosh-Bessel lens. S was found to be of the order of magnitude of 5 for the cosine-Bessel lens and much less for the cosh-Bessel lens. Let us take this figure of 5 as a trial value to probe the possibilities of reaching high resolving power with the unipotential lens, always with the restriction that this is only one special case and should not be looked at as necessarily indicative of the best lens that can be designed. From Eqs. (5) and (6)

$$\delta=(pS/2f^3)r^3. \tag{7}$$

For an objective lens whose focal length is 1 cm and which images at a sufficient distance to make p approximately equal to f, we find for the lens of $S=5$,

$$\delta=2.5r^3,$$

so that an aperture radius of about 5×10^{-3} cm would be small enough to insure 30A resolving power.

In computing S for the cosh or cosine potential distributions, the entrance and exit apertures were assumed to be small in diameter compared to lens length and, though the departure from exact cosine or cosh distribution due to these apertures was not accurately known, it was

possible to assign maximum values to the aberrations contributed by them. The coefficient B of Eq. (69) (appendix) is a measure of the spherical aberration and this equation may be looked at as stating that there is a limit to the value which B can attain for a lens of a given length. In particular, for the case of a very thin aperture lens (length of lens region small compared to its focal length), the voltage may be assumed substantially constant, and an approximate integration of (17) may be made. This point does not warrant further discussion here. It will probably be sufficient to state that for the two lenses considered the maximum values obtained for the additional spherical aberrations contributed by the apertures were computed to be too low to alter the order of magnitude of the figure previously given for S.

CHROMATIC ABERRATION

Formulas have been derived previously for chromatic aberration of electrostatic lenses.[8] A common formula is

$$\Delta f=(\Delta V/V)f, \tag{8}$$

where V is the voltage of the average electron and ΔV is the variation in voltage of electrons passing through the lens. It is quite apparent that this simple equation is quite inadequate for microscope lenses in which the voltage drop in the lens for all electrons is an appreciable fraction of the anode-cathode voltage. For this case it is convenient to think of the lens as consisting of a series of thin lenses, those near the entrance and exit being divergent and those near the lens center being convergent. The convergent action must, of course, be stronger than the divergent effects to yield a net convergent lens. But this converging action takes place at the lowest electron speeds. Thus in Eq. (8) one should use V_1 rather than V_0 for V for that portion of the lens. It was this effect that was mentioned previously as a limiting factor in attaining short focal lengths by utilizing large voltage drops.

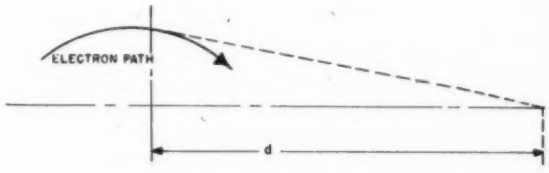

FIG. 13. Focal distance of electron.

For lenses in which the drop is near one-half V_0, a study of the cumulative action of the thin lenses assumed to make up the complete lens indicated that

$$\Delta f=(2\Delta V/V)f \tag{9}$$

will be a reasonably good approximation. Combining this equation with Eq. (5) yields

$$\delta=p/f(\Delta V/V)r.$$

The magnetic lens does not affect the magnitude of the velocity of the electrons. There is no occasion then to add a factor, such as the 2 of Eq. (9) to the aberration. For the same focal length and aperture radius, a magnetic lens may accordingly be expected to have less chromatic aberration than the electrostatic lens. The effect of this difference may be insignificant or important depending upon the complete instrument design.

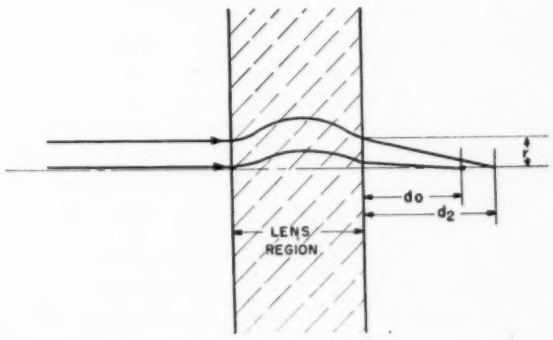

FIG. 14. Variation of focal distance for principal rays.

APPENDIX—SPREAD OF FOCAL LENGTHS DUE TO SPHERICAL ABERRATION

In the paper already referred to, Gray describes an electron's motion in an axially symmetric electrostatic field by specifying an instantaneous "focal distance" for each point of its path (Fig. 13). This distance d is the distance measured along the axis from the electron's position to the point on the axis toward which, or away from which, it is moving. With this definition and the usual equations of motion, the following approximate relation is obtained:

$$\frac{1}{d}=\frac{-A+Br^2-\cdots}{(2V)^{\frac{1}{2}}}, \tag{69}$$

where the coefficients A and B are determined by

$$(2V)^{\frac{1}{2}}A'+A^2=-\frac{V''}{2}, \tag{16}$$

$$(2V)^{\frac{1}{2}}B'+4AB=\frac{V''''}{16}-\frac{(A')^2}{2}, \tag{17}$$

the primes indicating differentiation with respect to the axial coordinate z and the equation numbers being Gray's.

Let us first consider the case of no aberration or

$$\frac{1}{d}=-\frac{A}{(2V)^{\frac{1}{2}}}, \tag{1a}$$

where A is determined completely by Eq. (16). If we choose a point on the axis in front of a lens of given potential distribution as a source of electrons (Fig. 14), we can plot any given electron's path through the lens by any of the well-known step-by-step methods.[7] One way, of course, is to start with d_1 and consequently [from Eq. (1a)] A_1 as known quantities at the lens entrance z_1, and compute A' and A step-by-step through the lens from Eq. (16). In this way we end up with A_2 at the lens exit z_2 and compute d_2. It is the variation of d_2 with r when B is included that gives the first approximation to the spherical aberration. If B_2, the value of B at the exit is known, the aberration can be computed, at least as to order of magnitude. Now B_1, the value of B at the entrance z_2 is zero, since the electrons all are specified to come from a single point on the axis, regardless of what their radial displacement is when entering the lens. Since the values of A and A' are known approximately from the first-order electron trajectory, we can now use Eq. (17) for a step-by-step computation of B.

An advantage of this technique of investigating spherical aberration over most other methods which have been described[8] is that it is possible to see how the spherical aberration is a result of distributed contributions from different parts of the lens region. The instantaneous value of the coefficient B is at any position along the lens a measure of the accumulated spherical aberration up to that point. It starts out with an initial value of zero. For a first approximation to freedom from spherical aberration, B should be zero or negligibly small at the lens exit. Notice that it is not necessary that it be zero at every point in the lens, so that the possibility of compensation between various portions of the lens is provided for. Taking a cue from light optics, we would expect that such compensation of the first disturbing term in the series [Eq. (69)] offers a possibility for improved lens systems.

With B_2 known, it is obviously possible to express the radius of the diffuse image of a single axial point. This Gray does in his Eq. (74). Such an expression would not tell the whole story with regard to the spherical aberration of a lens for use in an electron microscope. For this application, it is convenient to have an expression for the spread of focal lengths due to spherical aberration. There will be a band of focal lengths ranging from f to $f+\Delta f$, for the bundle of electrons leaving the lens at a distance from the axis ranging from zero to a radius of r. We choose an electron whose d at the lens entrance is $-\infty$, and whose d at the exit is therefore such that the electron crosses the axis at a focal point (Fig. 14). In the absence of spherical aberration B_2 is zero; we denote the exit d for this condition by d_0. Then

$$\frac{1}{d_0}-\frac{1}{d_2}=\frac{B_2r^2}{(2V_0)^{\frac{1}{2}}} \tag{2a}$$

or, approximately

$$\frac{d_2-d_0}{d_0^2}=\frac{B_2r^2}{(2V_0)^{\frac{1}{2}}}.$$

Since $d_2-d_0=\Delta f$

$$\Delta f=\frac{d_0^2B_2r^2}{(2V_0)^{\frac{1}{2}}}. \tag{3a}$$

For any given lens, Δf varies with the square of the lens stop radius. We now imagine the whole diagram of Fig. 14 to be magnified by some scale factor l, which increases all lens dimensions and the initial distance of the extreme ray from the axis in the same proportion. It is quite apparent that if the new electron paths through the new lens are to continue to be described by the equations we have written, then

$$l=\frac{d^*}{d}=\frac{A}{A^*}=\frac{Br^2}{B^*r^{*2}}$$

or

$$\frac{B^*}{B}=\frac{r^2}{lr^{*2}},$$

where the asterisk indicates the quantities after the scale change. Since r^{*2} is actually l^2r^2, we learn in this way that for a given lens geometry, B varies with inverse cube of any lens dimension. Since the focal distances and focal lengths are each proportional to the linear physical dimensions for any given lens geometry, and since V_0 is a constant independent of lens design, we may substitute in Eq. (3a)

$$d_0^2\propto f^2;\quad \frac{B_2}{(2V_0)^{\frac{1}{2}}}\propto\frac{1}{f^3} \tag{4a}$$

to obtain finally

$$\Delta f=Sr^2/f, \tag{5a}$$

where S is an aberration constant, characteristic of the particular type of lens. It is dimensionless, a function only of the lens' geometrical configuration. Its numerical magnitude can be found once B_2 has been found, of course, by comparing Eqs. (3a) and (4a).

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(8) L. M. Myers, *Electron Optics* (Chapman and Hall, Toronto, Canada, 1939).

Résumés of Recent Research

Precision Determination of h/e by Means of the Short Wave-Length Limit of the Continuous X-Ray Spectrum

Up to the present time there has existed a rather serious discrepancy among the values of the natural constants e, m, and h. This discrepancy has its most serious aspect in the failure of the "direct" x-ray determinations of h/e to agree satisfactorily with the value of h/e which can be obtained indirectly from measurements of the Avogadro number, e/m, and the Faraday.[1]

The precision x-ray method of measuring h/e hitherto used by all investigators depends on the determination of the Duane-Hunt limit by the method of isochromats. The method consists in monochromatizing an x-ray beam and then determining the intensity of the beam as a function of the accelerating voltage on the x-ray tube. In practice this method presents considerable difficulties owing to the following causes:

1. Imperfect monochromatization.
2. Small intensity available near the limit owing to the narrow pass-band of the monochromatizing device and insufficient primary intensity.
3. The difficulty of making reliable voltage measurements at high voltages to the required precision.
4. Voltage and current instability.
5. Interpretation of the isochromats as to the most probable position of the "true" limit.

In a recent paper Panofsky, Green, and DuMond[2] describe an experiment in which most of these difficulties are resolved satisfactorily.

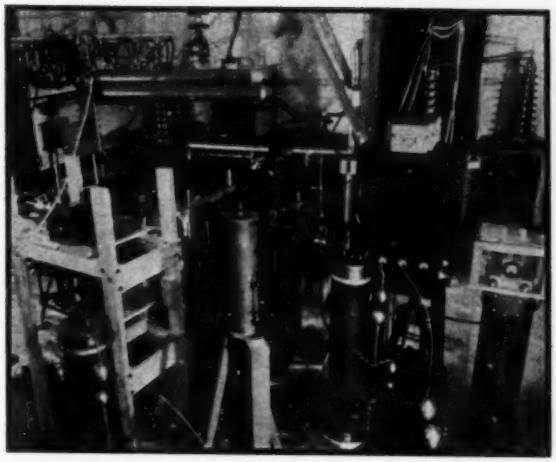

FIG. 1. Photograph of equipment used for the determination of h/e. In the direct fore-ground are the oil immersed precision voltage dividers. The wooden rack in front supports 1250 lb. of lead used to shield the ion chamber from stray radiation. The x-ray tube is directly behind the rack. The background shows the H. T. power supply and the vacuum system.

[1] R. T. Birge, Phys. Rev. **48**, 918 (1935); Rev. Mod. Phys. **13**, 233 (1941).

[2] Panofsky, Green, and DuMond, Phys. Rev. **62**, 214 (1942).

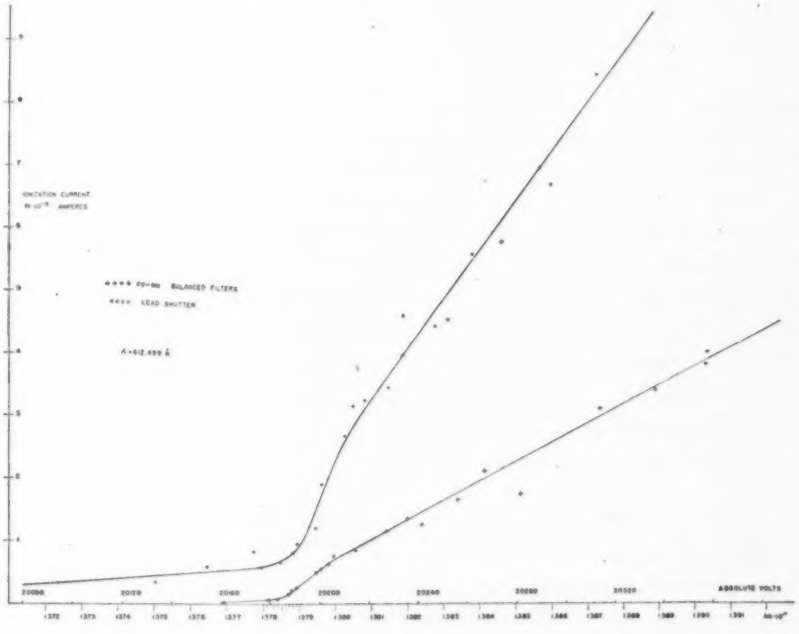

FIG. 2. Isochromats observed at $\lambda = 612.499$A. The upper curve is observed using a lead shutter for obtaining difference readings; the lower curve is obtained by using a set of Pd-Mo balanced filters. The corrected x-ray tube voltage is absolute volts and the computed h/e values are plotted on the abscissa.

Monochromatization of the x-ray beam was achieved by a precision 2-crystal spectrometer in conjunction with a set of balanced filters. Despite the narrow band of wave-length passed by this monochromatizing arrangement, sufficient intensity is available near the limit because of the large primary intensity employed. Power to the x-ray tube was supplied by a well-filtered half-wave power-supply capable of maintaining a current of 100 ma, a current 5 times greater than used previously in experiments of this type. A special voltage regulator of the degenerative type made it possible to maintain constant voltage throughout the experiment. Precautions for assuring the precision of the voltage measurements were taken by a special calibration procedure of the volt-box dividers used in conjunction with a potentiometer and standard cell for obtaining the voltage. It was found necessary to install a device in the x-ray tube to permit cleaning the target in vacuum. The use of this device eliminated errors due to thin layer deposits on the face of the target. Figure 1 shows the apparatus as used in the experiment.

Results showing graphically both the precision attainable by this method and the efficiency of the balanced filters used in conjunction with the spectrometer are shown in Fig. 2. The upper curve represents an isochromat taken with an x-ray beam monochromatized by the spectrometer only, while the lower curve represents an isochromat taken under identical condition to the upper one except for the addition of a set of balanced filters. (Space limitations forbid an explanation here of this "clipping" action of the balanced filters which greatly improves the interpretability of the isochromat. The reader is referred to the original article.[2]) A scale of h/e values is plotted on the abscissa of the graph, giving the resultant value of h/e for any intercept chosen. It was shown by an auxiliary analysis that the point of maximum bending of the isochromat ought to be interpreted as the true short wave-length limit. This procedure results in a value of

$$h/e = (1.3786 \pm 0.0002) \cdot 10^{-17} \text{ erg sec./e.s.u.}$$

The result is in much better agreement with the indirect values than previous values have been; the agreement is however not within the estimated probable error of the above result and of the "indirect value." The experiment makes it seem unlikely that there exist any fundamental flaws in the theories underlying the determination of h/e by the short wave-length limit method, as has hitherto been suggested several times.

New Studies on Extragalactic Nebulae Recent studies by Chandrasekhar,[1] Wyse and Mayall,[2] Zwicky,[3,4] and Hubble[5] have been concerned with (*a*) the morphology of nebulae and (*b*) the numbers of nebulae in relation to their luminosity.

Several attempts have been made to explain the structures of nebulae by an application of "particle mechanics" to the stars which supposedly constitute the major part of these nebulae.[1] This approach runs into two difficulties. First, because of the slow decrease of the gravitational forces with distance, the structure of nebulae depends essentially on *cooperative effects*, the mathematical analysis of which has not yet been satisfactorily mastered. Secondly, the nebulae, in addition to stars, contain vast amounts of *finely dispersed matter* whose influence on the structure of the nebulae is not properly understood. Comparison of theory with observations on some well-known extragalactic systems such as the Andromeda nebula and Messier 33 has resulted in some rather absurd conclusions, the most unsatisfactory of which perhaps is that the distribution of luminosity in a nebula appears unrelated to the density distribution.[2] Another approach to the problem has therefore been proposed which considers the nebulae in a first approximation as subject to the hydrodynamic laws in viscous gravitational fluids. After a phenomenological justification of the applicability of these laws, results on the distribution of mass, luminosity and average velocities in nebulae can be derived which are in satisfactory agreement with the known facts.[3]

Concerning the problem (*b*) mentioned, through

[1] S. Chandrasekhar, Astrophys. J. **90**, 1 (1939).
[2] A. B. Wyse and N. U. Mayall, Astrophys. J. **95**, 24 (1942).
[3] F. Zwicky, *Th. von Kármán Anniversary Volume* (1941), p. 137; Astrophys. J. **86**, 217 (1937) and **93**, 411 (1941).
[4] F. Zwicky, Phys. Rev. **61**, 489 (1942); Astrophys. J. **95**, 555 (1942).
[5] E. Hubble, Astrophys. J. **84**, 158 (1936).

extensive work with the large reflectors an empirical luminosity function for nebulae was derived which possesses a maximum for nebulae which are about seventy million times as bright as the sun.[5] Recent theoretical investigations, however, suggest that no such maximum should exist.[4] A systematic search for intrinsically faint nebulae in the local group of nebulae as well as in the Virgo cluster was therefore undertaken with the result that a number of interesting nearby faint nebulae such as the Leo system and the Sextans system were discovered. It was also found that a segregation of nebulae according to intrinsic brightness is associated with the formation of clusters of nebulae.[4] These observations lend support to the conclusions drawn from the theory and suggest that the luminosity function previously established is incorrect because of the interference of selective effects in the procedures used for sampling nebulae.

Here and There

Honors and Awards

Dr. Gustav Egloff, Director of Research of the Universal Oil Products Company, Chicago, was elected on September 4 an honorary member of the Chemical, Metallurgical, and Mining Society of South Africa.

The Chemical Industry Medal, one of the highest honors in the field of chemical science, was presented to Dr. Harrison E. Howe of Washington, D. C., for more than twenty years editor of *Industrial and Engineering Chemistry*. The presentation was made at a joint meeting of the New York Sections of the American Chemical Society, the Society of Chemical Industry, and the American Institute of Chemical Engineers, in New York City on November 6.

Dr. Peter Kapitza, Director of the Leningrad Physical Research Institute, was presented on October 27 with the Faraday Medal of the British Association of Electrical Engineers.

Dr. Arthur B. Lamb, Professor of Chemistry, Director of the Chemical Laboratory, and Dean of the Graduate School of Arts and Sciences at Harvard University, has been selected to receive the 1943 William H. Nichols Medal of the New York Section of the American Chemical Society.

Professor C. K. Leith of the University of Wisconsin is to receive the Penrose Medal of the Geological Society of America for 1942 at the Christmas meeting of this society to be held in New York.

Gerard Swope, President of the General Electric Company, has been selected the sixth recipient of the Hoover Medal, awarded by a joint committee of the American Society of Civil Engineers, American Institute of Mining and Metallurgical Engineers, the American Society of Mechanical Engineers, and the American Institute of Electrical Engineers. The presentation is to be made at the winter convention of the American Institute of Electrical Engineers during the week beginning January 25.

The John Fritz Medal for "notable scientific or industria achievement" has been awarded to Dr. Willis Rodney Whitney, honorary Vice President of the General Electric Company. The Board of Award consists of representatives of the four engineering societies which also administer the Hoover Medal.

The presentation of the Perkin Medal of the Society of Chemical Industry to Dr. Robert E. Wilson, President of the Pan-American Petroleum and Transport Company and of the American Oil Company, will take place on January 8 at a meeting of the society in New York.

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Recent Appointments

Dr. William T. Anderson, Jr., for nearly twenty years Director of the Radiation Research Laboratory of the Hanovia Chemical and Manufacturing Company, Newark, New Jersey, has been granted leave of absence to accept a commission as Lieutenant in the U. S. Naval Reserve.

Dr. R. V. Churchill has been promoted to a full professorship in the Department of Mathematics at the University of Michigan. In the same department, Dr. P. S. Dwyer is now associate professor and Dr. R. M. Thrall, assistant professor.

Dr. E. R. Gilliland, Professor of Chemical Engineering at Massachusetts Institute of Technology, and Ray P. Dinsmore of the Goodyear Tire and Rubber Company, Akron, Ohio, are among the consultants appointed by the rubber director, William M. Jeffers, to study the technical aspects of the rubber problem.

Professor Frederick Seitz has resigned from the faculty of the Department of Physics, University of Pennsylvania, to become Chairman of the Department of Physics at Carnegie Institute of Technology.

Dr. Alfred Tarski, Polish mathematician and logician, formerly of the University of Warsaw, has joined the faculty of the University of California at Berkeley for the duration of the war.

Society News

Professor C. H. Mathewson, Chairman of the Department of Metallurgy at Yale University, has been elected President of the American Institute of Mining and Metallurgical Engineers. Erle V. Daveler, Vice President of the Utah Copper Company, and Harvey S. Mudd, consulting engineer of Los Angeles, were elected vice presidents.

At the annual meeting of the Physical Society, London, on October 2 the following officers were elected for the year 1942–43: President, Sir Charles Darwin; New Vice Presidents, Professor C. D. Ellis and Dr. H. T. Flint; Honorary Treasurer, Dr. C. C. Paterson; Honorary Secretaries, J. H. Awbery (papers) and Dr. W. Jevons (business); Honorary Foreign Secretary, Sir Owen Richardson; Honorary Librarian, Professor L. C. Martin; New Members of Council, E. R. Davies, Dr. W. B. Mann, A. J. Philpot, Professor H. C. Webster, and Dr. W. D. Wright.

The Physical Society has founded a biennial Rutherford Memorial Lecture. The first lecture was delivered by Professor H. R. Robinson, Professor of Physics in the Queen Mary College, University of London, on the subject of Rutherford's early life and work.

The meeting of the American Physical Society which was originally scheduled to be held December 28–30, in conjunction with the American Association for the Advancement of Science, has been postponed to Friday and Saturday, January 22 and 23, 1943. The postponed meeting will be held at Columbia University, New York City, and the program will include all papers submitted for the original meeting. The American Association of Physics Teachers and the American Society for X-Ray and Electron Diffraction will meet at the same time. It is announced that the meeting of the Physical Society normally held in February will be omitted in 1943.

The annual meeting of the American Association of Petroleum Geologists will be held at Fort Worth, Texas, on April 7, 8, and 9. All meetings, including committee, research, and business meetings, will be limited to these three days.

The Institute of Radio Engineers will hold its National Winter Conference on January 28, 1943 in the Engineering Societies Building, 33 West 39 Street, New York City. This one-day conference replaces the three-day conference normally held during January. Features will include the annual meeting of the Institute, induction of officers for the year 1943, presentation of the medal of honor, award of Fellowship and other awards. The day's activities will culminate in a joint A.I.E.E.-I.R.E. meeting with an address on the subject of ultra-high frequencies.

Dr. J. D. Bernal, Professor of Physics at Birkbeck College, University of London, delivered at the Royal Institution on November 19 the first Sir William Bragg Memorial Lecture of the Chemical Society, London.

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Ohio State Radiation Laboratory

Ground has been broken on the engineering quadrangle of Ohio State University for a small laboratory building to comprise two new units of the radiation laboratory. When completed about March 1, this building will house an electrostatic generator developing three-million-volt x-rays, used to produce artificial radioactive substances, and an electron accelerator producing twenty-million-volt rays. The new laboratory is to have double walls of concrete blocks with earth between, is to be constructed half above ground and half below, and will have a concrete roof. Other units of the radiation laboratory already installed are a cyclotron in the Engineering Experiment Station and an electron microscope in the Communications Laboratory. The radiation laboratory represents a cooperative research program of the departments of physics, electrical engineering, chemistry, and medicine, although the equipment is also available to other university departments.

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Westinghouse Research Fellowships

Because both research and production are being devoted almost wholly to the war effort, the Westinghouse Electric and Manufacturing Company has discontinued for the duration its post-doctorate research fellowships which have been awarded annually for the past five years. It is expected that those who in normal times would be interested in research scholarships will be directed to research work of war value.

★

Necrology

Dr. Albert L. Barrows, Executive Secretary of the National Research Council, died on November 7 at the age of fifty-nine.

Dr. Frederick M. Becket, well-known metallurgist and former Vice President of Union Carbide and Carbon Corporation, died on December 1 at the age of sixty-seven.

Nelson J. Darling, manager of the plants of the General Electric Company at Lynn and Everett, Massachusetts, died on October 26 at the age of fifty-eight.

Dr. Henry Gordon Gale, Professor of Physics and Dean Emeritus of the Division of Physical Sciences of the University of Chicago, died on November 16 at the age of sixty-eight.

Dr. Charles Nelson Haskins, Chandler Professor of Mathematics at Dartmouth College, died on November 14 at the age of sixty-eight.

Dr. Leonard Klatzow, member of the research laboratories of Electrical and Musical Industries, Ltd., died on September 22 at the age of thirty-five.

Professor G. H. Ling, formerly Professor of Mathematics in the University of Saskatchewan, died recently at the age of sixty-eight.

Dr. Robert Linton, consulting mining and industrial engineer of Los Angeles, died on November 12 at the age of seventy-two.

George Burr Upton, Professor of Automotive Engineering at Cornell University, died on October 2 at the age of sixty.

Contributed Original Research

Fine Structure of Metallic Surfaces with the Electron Microscope

R. D. HEIDENREICH AND V. G. PECK
The Dow Chemical Company, Midland, Michigan
(Received August 28, 1942)

Several techniques for producing surface replicas to be used in the transmission electron microscope are briefly discussed and compared on the basis of experience with them in this laboratory. A new two-step replica process is described which utilizes molded polystyrene as the first replica and evaporated silica as the second, thin film replica. The high mobility of condensing silica vapor on polystyrene is pointed out and it is shown that such evaporated films tend to fill in small surface irregularities and so produce a replica of fine surface structure which appears to be reliable. A table is included which gives an idea of the amounts of silica required for several surfaces of different "roughness" to yield films that are of about the same quality for use in the electron microscope. Electron micrographs of replicas so prepared are exhibited illustrating some pearlitic structures in a 0.98 C, hot roll steel, precipitation in a Cu-Be alloy, and a calcite cleavage plane. Excellent resolution can be obtained in such replicas and a resolution of approximately 50A is demonstrated in the calcite cleavage plane.

INTRODUCTION

THE application of the electron microscope to surfaces of opaque bodies has become a major problem of increasing importance in studies of the structure of alloys, corrosion phenomena, etc. Several methods are under development by various workers and fall into two categories: the direct observation of surface structure by a suitable electron optical system* and the reproduction of the surface structure into a thin film of material suitable for use in the transmission type electron microscope. These two methods offer quite different problems for which only partial solutions have been achieved in either case. It is the reproduction technique that is to be considered in this paper.

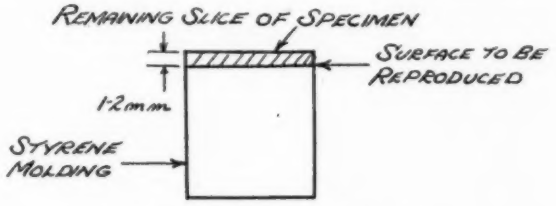

FIG. 1. Polystyrene replica.

Several techniques of obtaining thin film replicas of surfaces have appeared in the literature and can be placed in two main types or classes. These are: (1) Natural surface films and (2) Synthetic surface films.

1. Natural Surface Films

This replica method consists in producing a thin, structureless film, generally oxide, on the previously prepared surface and then removing the film by appropriate chemical means. This technique has been used by Mahl[1–3] and applied to aluminum and nickel with considerable success. Its application is limited because of the requirement that the film be structureless to the electron microscope and that it can be removed from the surface without physical or chemical damage. Metals that tend to form natural protective films are, of course, best suited to this technique.

2. Synthetic Surface Films

This class of replicas has appeared as both one-step and two-step processes.

* For a discussion of such methods, see: V. K. Zworykin, J. Hillier, and R. L. Snyder, A.S.T.M. Bull. No. 117, pp. 15–23 (August, 1942).

[1] H. Mahl, Zeits. f. tech. Physik **21**, 17 (1940).
[2] H. Mahl, Metallwirtshaft **19**, 1082 (1940).
[3] H. Mahl, Zeit. f. tech. Physik **22**, 33 (1941).

FIG. 2. Grinding marks on a steel surface finished on a chamois wheel with levigated alumina.

a. One-Step, Direct Replicas

The direct replica technique consists in casting a thin film of plastic on the prepared surface from a suitable solvent and, after it has set, stripping it from the surface with the aid of a liquid medium such as water. This method is being used by several workers, chief among whom are Schaefer and Harker.[4] It has the advantages of speed and preservation of the specimen but leaves something to be desired in contrast and resolution. In interpreting this type of replica, high spots on the original surface appear light in the micrograph.

b. Two-Step, Positive Replicas

The two-step process has as its aim the elimination of mechanical stripping of the thin films which may prove difficult on some surfaces. This method consists in first reproducing the prepared surface into a bulk material which can be easily removed from the original and then making a thin film replica of the first replica. The thin film is removed from the first replica by chemical means. The silver-collodion process of Zworykin and Ramberg[5] has been the chief two-step method up to this time.

The silver-collodion technique seems to give better resolution than the direct replicas but is limited in application by the tendency of the silver to alloy with some of the common metals. Interpretation is somewhat complicated by the residual silver[6] left in the plastic film.

[4] V. J. Schaefer and D. Harker, J. App. Phys. **13**, 427 (1942).

[5] V. K. Zworykin and E. G. Ramberg, J. App. Phys. **12** 642 (1941).

POLYSTYRENE-SILICA REPLICAS

The technique developed in these laboratories is a two-step process in which the first replica is of molded polystyrene and the second, thin film replica is evaporated silica. The formation of this molded plastic replica is simple in principle, consisting of simply making an impression of the original surface in a material which can be readily handled in the remainder of the process. The second replica which is observed in the microscope was discovered by chance since the property which makes it suitable for this purpose was not predicted. This property is the high mobility of silica condensing from the vapor both upon polystyrene and upon the silica in its vitreous state. If silica be evaporated onto a block of styrene, it is found upon removal of the film in a solvent that the film completely surrounds the styrene. This was well demonstrated by placing the styrene block in an enclosure having a small aperture (about 0.003 in.) facing the vapor source and carrying out the evaporation. Instead of a spot of silica film on the front

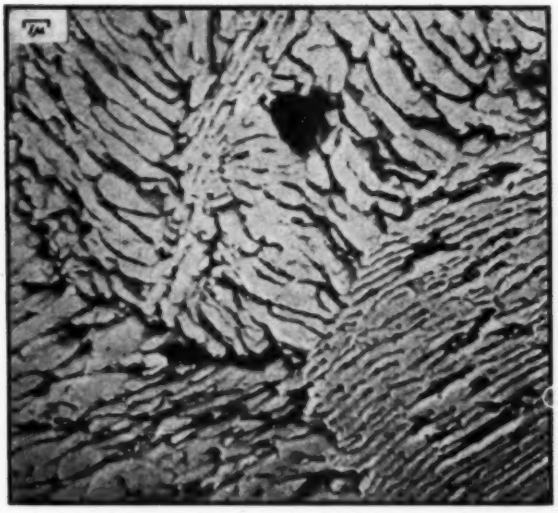

FIG. 3. 0.98 C, hot roll steel with mechanical polish and nital etch.

[6] R. F. Mehl, Metal Progress **40**, 759 (1941).

face of the styrene, it was found that the entire block was again covered.

When a heavy evaporated silica film is removed from a styrene replica of an etched metal surface and examined under the light microscope with vertical illumination, interference colors are observed with abrupt changes in color, indicating changes in thickness. Films from a smooth styrene surface (fire-polished) exhibit a uniform appearance, thus indicating the surface topography of the styrene replica has imparted variations in thickness to the silica film, at least partially corresponding to the original surface. This is further borne out by Table I in which are

TABLE I. Weights of silica per unit area to give comparable replicas.

Surface	$\omega = \frac{\text{micrograms}}{\text{cm}^2}$	Average film thickness referred to plane surface
Cellulose lacquer film	5	200A
Styrene moldings		
Polished steel. No etch	10	400
Etch Cu-Be	12	500
Etched steel	28	1300

tabulated the weights of silica per unit area of styrene that yield replica that are comparable in the electron microscope. It is evident that the more structure the surface presents the more silica is required to produce a satisfactory replica.

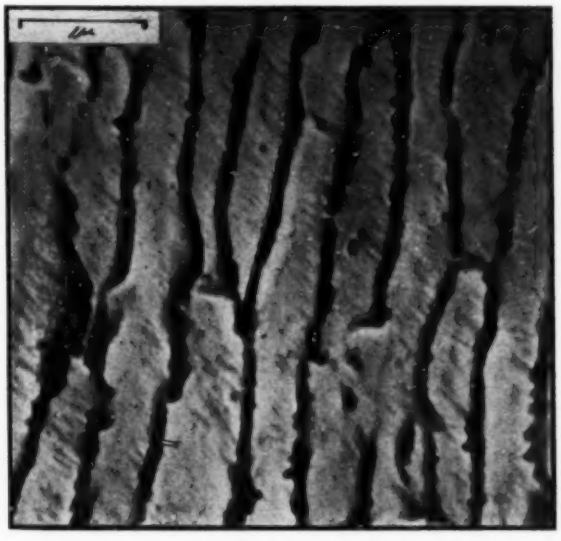

FIG. 4. Lamellar pearlite in 0.98 C, hot roll steel.

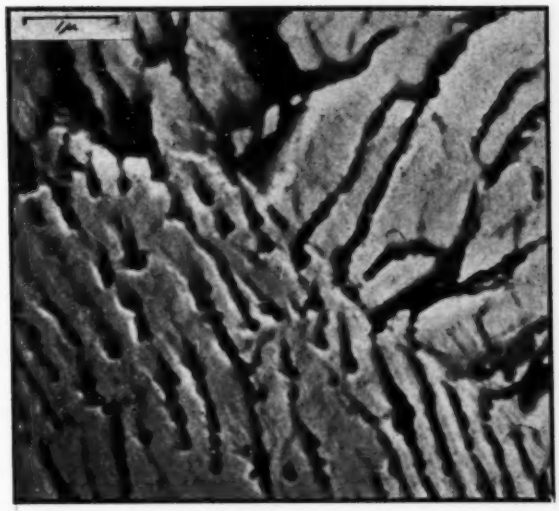

FIG. 5. Partially spherodized pearlite in 0.98 C, hot roll steel.

It is concluded, then, that the top surface of the silica film is flat to a certain degree although to just what degree is not yet determined. It is estimated, however, that the surface irregularities exceeding about 0.1μ variation normal to the surface no longer yield a level top surface in the films used for electron microscope study. It is expected that coarse structure will be reproduced in a manner similar to that of natural oxide films and this fact must be taken into account in the interpretation.[2,3] Investigations into this matter are underway. The fact remains, however, that the fine structure that is of interest in electron microscope studies is reproduced with a considerable degree of faithfulness.

The silica films are themselves structureless to the electron microscope and electron diffraction shows them to be amorphous. They are easily removed from the styrene by the use of ethyl bromide solvent and washed free of any residual styrene. It is further found that thin films (100–200A) removed in ethyl bromide and mounted on specimen screens can be heated to 1000°C in vacuum and at least 500°C in air with no apparent effect as far as the electron microscope is concerned. Thus they may be used as a temperature-stable support for electron microscope specimens when formed on an "electron microscopically plane" surface such as a cast plastic film. Their chief disadvantage is the fact

that, if the replica films are much too thick, they charge up badly and break under the electron beam. Careful control of the evaporation will yield films, however, which are completely satisfactory once the amount of silica for a given surface has been determined.

TECHNIQUE

The specimen size for making such replicas is quite arbitrary, but a sample of 0.5-cm thickness and about 1.5-cm square is convenient to work with. Thin specimens may be used as well if they are mounted in polystyrene and polished in the usual manner. The polish and etch are dependent upon the metal and structure to be studied. The steps in the process are as follows.

1. Molding

Immediately after preparing the surface, polystyrene is molded onto it, since the time delay after etching should be a minimum to reduce effects of air and water vapor. A low viscosity polystyrene in the form of granules and dust-free is suitable. An ordinary mounting press will serve for making the molding. The specimen, mounted or unmounted, is placed in the press with sufficient polystyrene granules on the surface to give a final molding of a thickness of about ½ inch. The mold is then heated to 130°C before applying any pressure since premature application of pressure will press the hard granules against the surface and result in deformation. This is particularly true for soft metals. A pressure of 2000–5000 pounds is applied and the temperature raised to 160°C. The mold is then allowed to cool while maintaining constant pressure until the temperature has dropped to well below 80°C. This precaution is necessary to prevent formation of bubbles in the molding.

FIG. 6. 0.98 C, hot roll steel with electrolytic polish and nital etch.

The removal of the metal from the polystyrene is accomplished by first carefully and slowly sawing off the excess polystyrene and metal from the back side of the polished surface to leave a thin slice of metal about 1–2 mm thick imbedded in the polystyrene. The excess polystyrene around the periphery of the specimen is next removed so that only the desired surface will yield replica film. The polystyrene molding and thin metal slice now appear as shown in Fig. 1.

In some cases the metal can be removed by a sharp tap if so desired, but more often must be dissolved in acid. Any of the mineral acids (except H_2SO_4) in moderate concentration (1 : 3) can be safely used without damage to the polystyrene. Acetic acid and caustic may be used as well. When the metal is all dissolved, the polystyrene surface is washed with dilute acid followed by a swift stream of distilled water and then dried in clean air. It should then be examined in a light microscope for any dirt or salt that might remain on the surface.

2. Silica Evaporation

Silica is now evaporated on the styrene replica surface from a conical tungsten filament wound from 20-mil tungsten wire. A convenient size is a cone about 1 cm long and an angle of 30° with the apex sharp enough to prevent the small silica splinters from falling through.

A known weight of high purity quartz in the form of small splinters (splinters weighing less than 1 milligram) is placed in the apex of the filament and the styrene molding mounted vertically above it at a distance of 6–8 cm. A current of 18–20 amp. a.c. will evaporate the quartz in about 30 sec. without causing the temperature of the styrene surface to rise above 50°C. A vacuum of approximately 10^{-4} mm of

FIG. 7. 0.98 C, hot roll steel with electrolytic polish and nital etch.

Hg or better is advisable to avoid difficulties with oxides of tungsten. The amount of quartz to be evaporated will depend upon the roughness of the surfaces and can be approximately determined by

$$\omega = 0.3(\bar{W}/R^2) = \text{weight of silica per cm}^2, \quad (1)$$

where $\bar{W}$ is the weight of quartz in grams and R is the distance from apex to surface in cm; $\bar{W}=3$ milligrams and $R=6$ cm are satisfactory values for etched, pearlitic steel. Equation (1) is valid only for polystyrene due to the extreme mobility of condensing silica on this surface.

The back and sides of the styrene molding will be covered with film as well as the front and this should be removed with fine emery cloth. The surface is then cut into squares about $\frac{1}{8}$ in. on a side, using a sharp knife, and the whole placed in ethyl bromide, replica side up. A shallow dish is used with sufficient solvent to cover the surface of the molding when it is pushed to the bottom where it will stick after being wet for a few seconds. The silica squares will be released from the styrene within 5 minutes and can easily be seen in the liquid where they are constantly in motion. Side illumination and a piece of red glass under the dish greatly aid in seeing the films. The films are removed from the first solvent with a piece of fine screen and placed in a dish of fresh solvent to remove any adhering styrene. They are picked up from here on the usual specimen screen and are ready for the microscope. If the films tend to roll up, they generally can be "teased" until they unroll on the screen.

RESULTS

The technique just described has been applied with good results to some 75 specimens and more than 200 pictures have been taken. The method is limited to materials that can withstand the molding pressure and temperature. Complications due to oxide films may arise on some metals, but in general this has not proven too serious. In such cases it would seem better to use the oxide film itself as the replica as described in references 2 and 3. For surface preparation, the common metallographic techniques give good results on most metals. Etchants which leave a residual film on the surface are to be avoided.

The following examples of this replica technique were made with an RCA electron microscope, type B. Figure 2 is a silica replica of a steel surface finished on a chamois wheel using levigated alumina. The scratches are quite evident and, in all cases, a high ridge of metal is

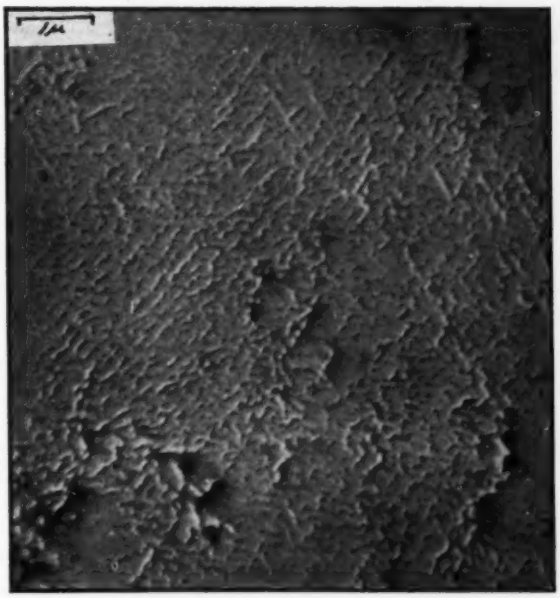

FIG. 8. Precipitation in a Cu-Be alloy.

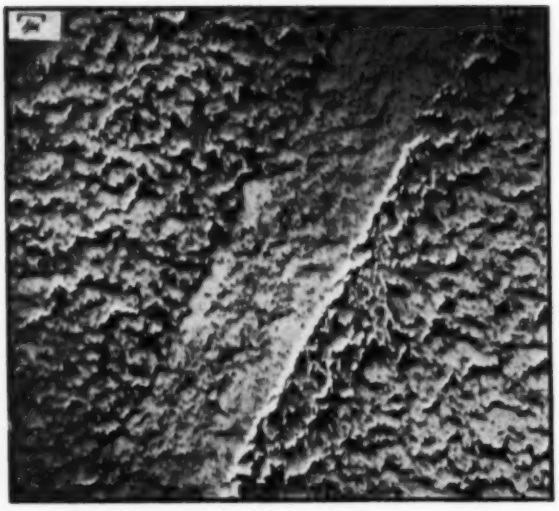

FIG. 9. Twinning in Cu-Be alloy.

found on one or both sides of the groove. The high spots on the original surface here appear dark.

Figures 3, 4, and 5 are micrographs of a silica replica made from 0.98 C, hot roll steel given a mechanical polish and a nital* etch. Figure 3 is a low magnification picture (taken at 3300 ×) showing portions of three grains and illustrating variations in the types of pearlitic structure that can be observed in this specimen. An interpretation of these pictures is not the purpose of this paper since they are merely presented as examples of what can be done with the technique and as a demonstration of the ability of evaporated silica to produce a replica which appears entirely reasonable.

Figure 4 is a high magnification micrograph of relatively well-developed pearlite as compared with that seen in Fig. 5. Figure 5 was taken in a region quite similar to the boundary of two of the grains in Fig. 3 and probably is an example of pearlite which is either partially spherodized or was never well-formed. What appear to be broken down lamellae have been observed in other regions of this replica. Varying degrees of this phenomenon were found in different grains.

This same steel was also given an electrolytic polish and a nital etch. The silica replica of this surface yielded several very interesting isolated spots. Figure 6 could be interpreted as a field of small lamellae lying nearly horizontal with the surface. The resolution and detail obtainable are well-illustrated here. Figure 7 shows some rather unusual right-angle structure which was found in one region of the replica. Some discussion concerning the interpretation of this micrograph has arisen as to whether the black lines are carbide lamellae with ferrite between them or whether these are carbide lamellae lying approximately horizontal to the surface. Inasmuch as the lamellae in other micrographs are generally not spaced closer than 0.5μ, it is possible that, if they are oriented horizontal to the surface, they constitute a "step" structure which is too "rough" for the film thickness employed in this replica and might so yield the micrograph of Fig. 7. Consequently it can only be said with any certainty that the cubic crystal habit of one of the phases is illustrated in Fig. 7 with a detailed interpretation reserved until final results are obtained in an investigation of the silica replicas now in progress.

An example of the ability of the technique to reproduce fine structure is shown in Fig. 8, which is a micrograph of a Cu-Be alloy.** Precipitation of small particles, about 500A, along the atomic planes is illustrated.

Figure 9 illustrates twinned structure in this same Cu-Be alloy considerable of which was observed in the specimen with the light microscope.

The light line along one side of Fig. 9 is probably due to mechanical breakage of the film at the sharp angle where high stress concentration would arise. This is observed in fields having sharp thickness variations.

As a final example of polystyrene-silica replicas, Fig. 10 illustrates the resolution that can be obtained. This replica was prepared from a freshly cleaved calcite crystal which suffered rather drastic deformation during molding and hence exhibited considerable twinning and slip. Figure 10 was taken from a band of what are presumably slip lines in a deformed region of the surface. Examination of this micrograph will indicate that distances of less than 100A are

* Two percent nitric acid in ethanol.

** Alloy obtained from General Electric Company and reproduced with their permission.

easily observed and that a resolution of about 40A has been obtained.

GENERAL REMARKS

The replica method just described leaves something to be desired as far as the time required to make a reproduction (about 2 hours from polishing room to microscope) is concerned and as to simplicity. Good contrast and resolution are easily obtainable, however, and the replicas are quite reliable from what is known at present. Investigations are under way to determine exactly how the replicas are to be interpreted, both for coarse and fine structure. It is quite certain that the reproduction of fine structure is to be interpreted on the basis of an almost level top surface on the silica film. Figure 7, however, is an example of the sort of problems that will arise in interpretation and points to the use of stereomicrographs and microphotometer traces in detailed analysis of replicas.

It becomes more evident that no single method will fulfill all the requirements of a replica, but rather that a combination of results with the different methods must be employed. The technique most likely to best bring out an anticipated structure must be chosen, which in turn means that the several techniques must be applied to the same surface and then evaluated by careful comparison.

In conclusion, it should be mentioned that modifications of this molding technique are possible. In fact, the first attempts at this sort of a replica employed polyvinyl alcohol (cold water soluble) as the first replica and Formvar or ethyl cellulose as the second replica. This was quite similar to the silver-collodion technique with the molded polyvinyl alcohol substituted for the silver. It was abandoned chiefly because of the chemical attack of the metal by water present in the polyvinyl alcohol.

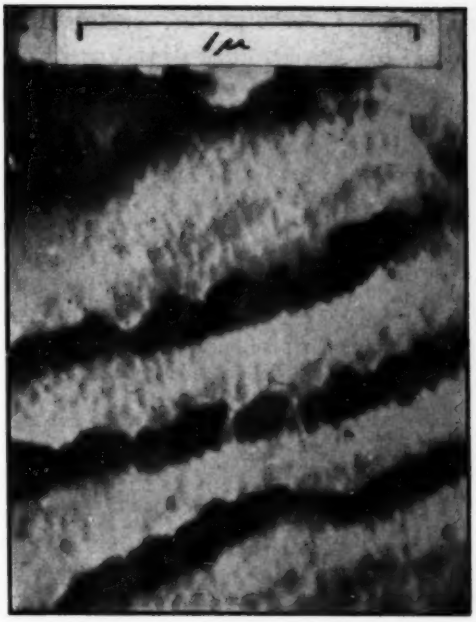

FIG. 10. Slip bands in a calcite crystal.

Plastics other than polystyrene can be used in this process (such as methyl methacrylate), but polystyrene offers the advantages of high dimensional stability, low moisture absorption, and chemical inertness. Aluminum oxide can be substituted for silica, but here again there is no apparent advantage since both evaporate at the same temperature and are very similar as far as electrical properties are concerned. For special purposes, it is quite probable that modifications of the present technique will arise.

The Use of a Mechanical Synthesizer to Solve Trigonometric and Certain Types of Transcendental Equations, and for the Double Summations Involved in Patterson Contours

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A method is shown whereby equations in trigonometric form and types of transcendental equations, which can be expanded into rapidly converging series, may be solved by a mechanical and graphical process. A mechanical and graphical method is also shown which effects the double summation of Fourier series used in the Patterson method for the determination of interatomic distances in crystals. Graphs are given to illustrate each of the processes discussed.

A mechanical synthesizer[1] can be used to solve equations expressed in trigonometric form and for equations containing transcendental functions which can be expanded into rapidly converging series (Fig. 1). If an equation is trigonometric in form and composed only of sine and cosine terms, it may be set up directly upon the machine (powers are expanded into terms containing multiple angles). The coefficients of each harmonic term, after adoption of a suitable scale, are the magnitudes of the setting of the corresponding harmonic elements of the machine. The angle is caused to vary from zero to 2π, and a graph is made of the value of the function against the angle. A crossing of the axis by the curve signifies a zero value of the function, and the corresponding value of θ is a root value of the equation.

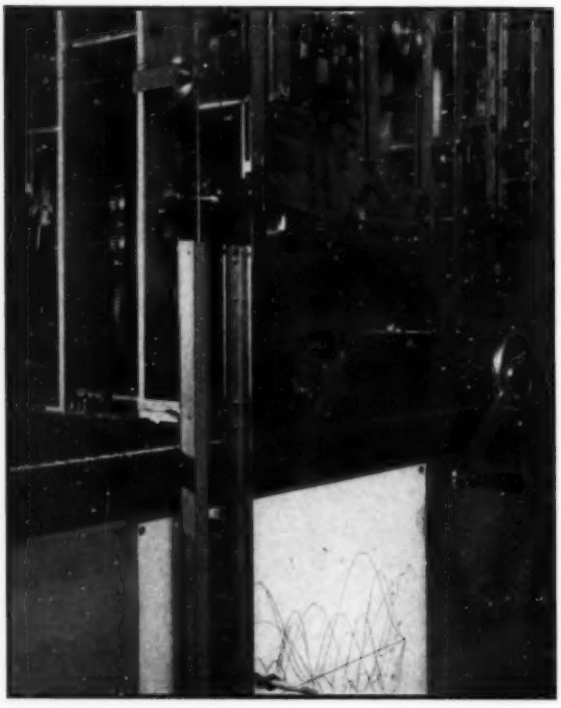

Fig. 1. A view of the synthesizer.

If, as in the equation in Fig. 2, the trigonometric equation contains discontinuous terms such as $\tan\theta$ ($\cot\theta$, $\sec\theta$, $\csc\theta$), the equation is transformed by multiplication by $\cos\theta$ ($\sin\theta$, $\cos\theta$, $\sin\theta$), thus giving an equation containing sine and cosine terms only. This transformed equation has the same root values as the original equation and can be solved by the method previously mentioned.

In Fig. 3 and Fig. 4 are shown mechanically graphed Légendre polynomials and associated Légendre functions.

As an example of the mechanical solution for a type of equation of the form $x+\sin x=$ constant, one may consider the elliptical path of motion of one body moving about another body. For small eccentricity (e) of the ellipse, the polar angle $\Phi=\omega t+2e\sin\omega t$, where t is the time measured from the instant when the particle describing the ellipse (e.g., under the gravitational attraction of a massive particle situated at the focus of the ellipse) passes the end of the major axis, and ω is 2π times the frequency. For a value of $e=0.5$, the equation used to determine the time elapsing until Φ has the value of 3 radians is $3=\omega t+\sin\omega t$ or $3=x+\sin x$ (Fig. 5). To solve for x (ωt) one can expand $\sin x$

[1] S. Leroy Brown, "A mechanical harmonic synthesizer-analyzer," J. Frank. Inst. **228**, No. 6 (December, 1939).

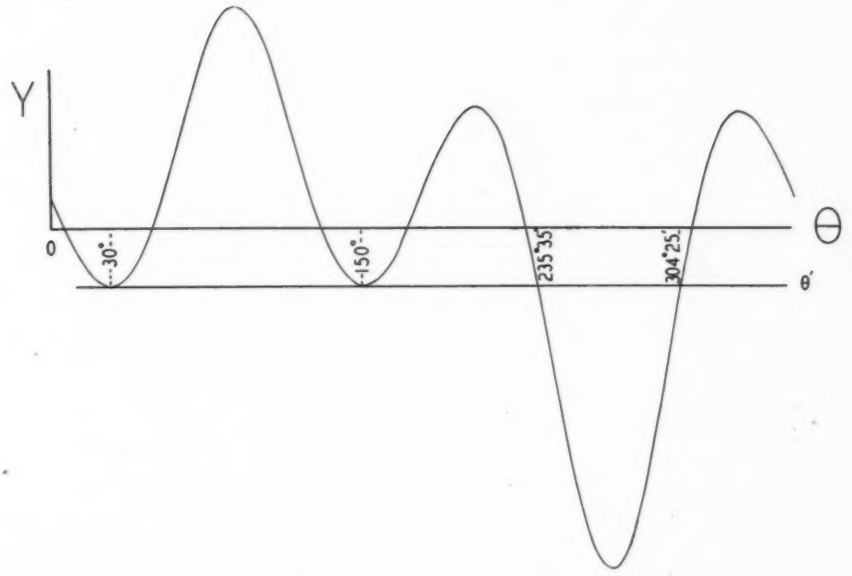

FIG. 2. $3 \tan^2\theta + 2\cos^2\theta + \sin\theta + \csc\theta = 5 = Y + 5$.

in a power series obtaining for the equation

$$3 = x + x - (x^3/3!) + x^5/5!) - (x^7/7!) + \cdots. \quad (1)$$

Making the transformation $x = 2.5\cos\theta$, Eq. (1) becomes

$$3 = 3.489\cos\theta - 0.436\cos 3\theta + 0.0376\cos 5\theta - 0.00189\cos 7\theta + \cdots. \quad (2)$$

The synthesizer used to graph the right-hand member of Eq. (2) possesses fifteen harmonic elements. For this equation four terms were used since the effect of the fifth term, due to the small size of its coefficient, will not appreciably affect the accuracy of the result. The curve representing the right-hand member of Eq. (2) crosses the curve-axis θ' (determined by the constant term of the equation) at $\theta = 29°$. This gives for x (or ωt) a value of 2.187 radians.

Consideration of a special case of a suspended cable results in a problem that is also adapted to mechanical solution. If the load per unit length of a cable is w, the separation of two supports

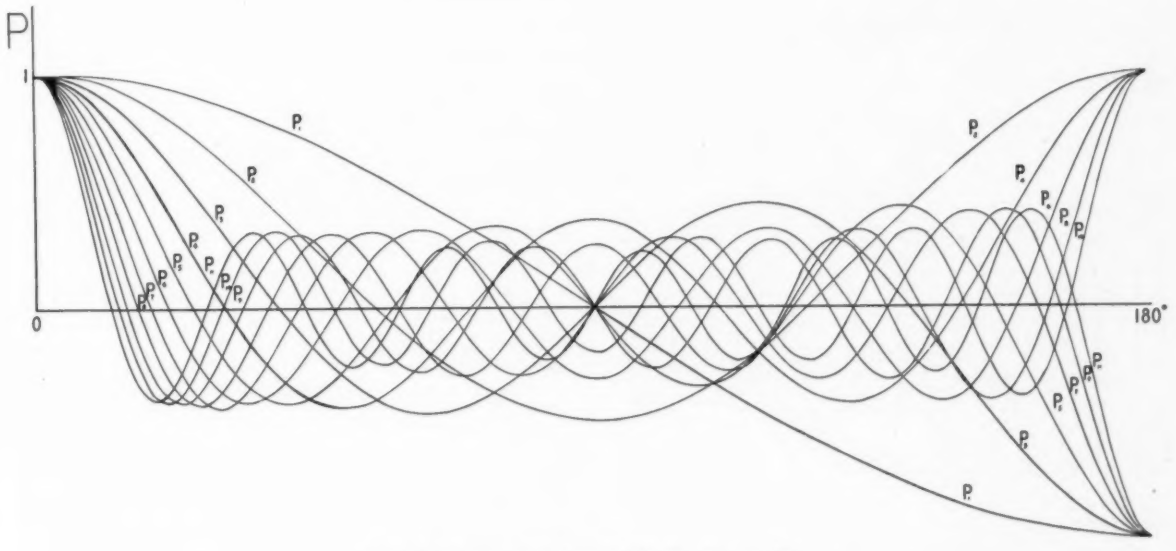

FIG. 3. Légendre polynomials $P_1, P_2, \cdots P_{11}$.

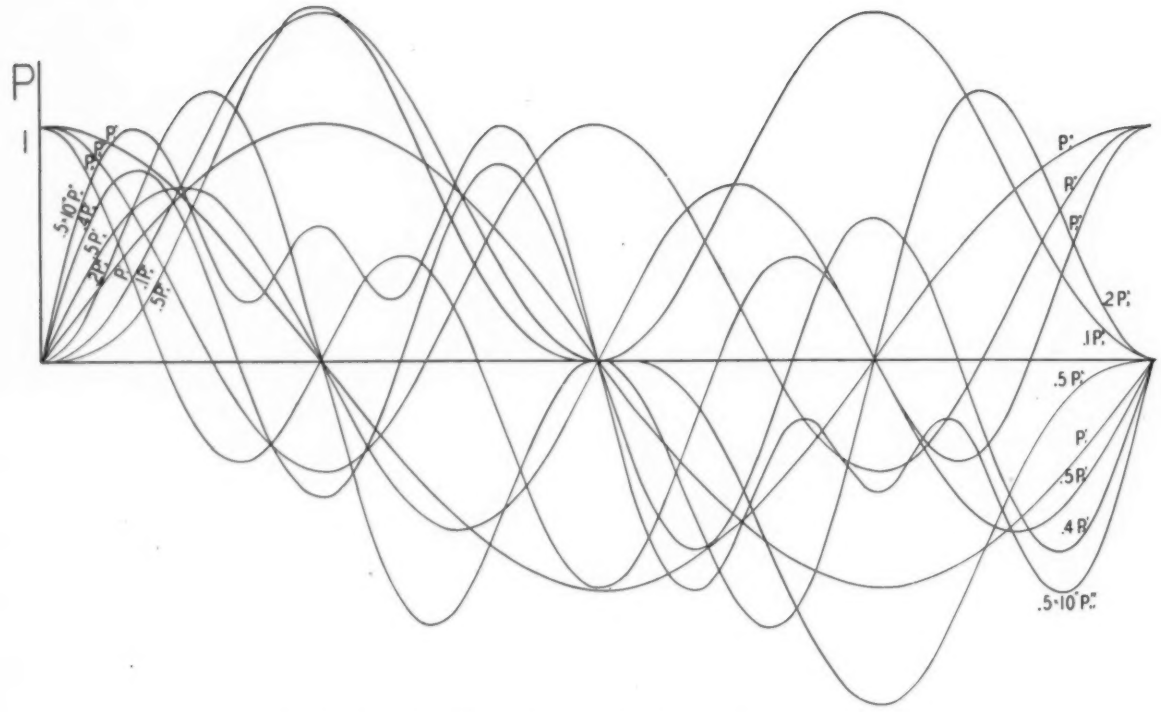

FIG. 4. Associated Légendre functions P_1^0, $P_1^1 \cdots P_3^3$ and P_{11}^{11}.

of equal height is b, the sag is h, and the tension of the cable at its middle point is P, then for the stretched cable one may express the relation between the sag h and the tension P as follows:

$$h=(P/2w)(e^{wb/2P}+e^{-wb/2P}-2). \qquad (3)$$

For $w=2$ lb./ft., $h=3$ ft., and $b=6$ ft.,

$$6/P+1=\cosh 6/P, \qquad (4)$$

and letting $x=6/P$,

$$x+1=\cosh x. \qquad (5)$$

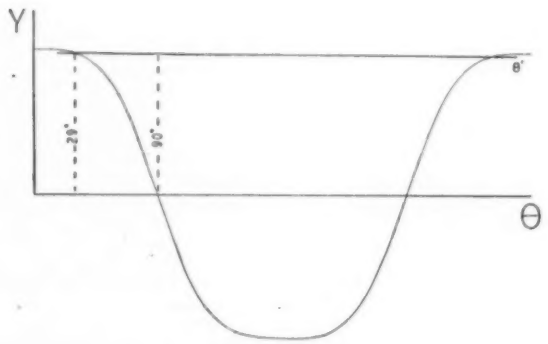

FIG. 5. $X+\sin X=3=Y+3$.

Expanding $\cosh x$, and making the transformation $x=2\cos\theta$, Eq. (5) becomes

$$-7.67=-12\cos\theta+8.25\cos 2\theta +0.6\cos 4\theta+0.016\cos 6\theta+\cdots. \qquad (6)$$

The neglected terms are sufficiently small to be omitted in the machine solution. Figure 6 shows the resulting graph.

Two root values for θ were found, $\theta=35°\ 45'$ and $\theta=90°$, or $P=6/1.623$ lb. and $P=6/0$ lb. The root value of P is, therefore, 3.70 lb. for θ equal to $35°\ 45'$, and it is infinite for θ equal to $90°$.

The diagram used in solving the equation

$$\log(2x^3+5x^2+10x+15)=5/x \qquad (7)$$

is shown in Fig. 7. In this case the equation used was

$$2x^3+5x^2+10x+15=e^{5/x}.$$

The right member ($e^{5/x}$) was expanded into a series and the entire equation then multiplied by the highest power of x retained in the right member. Substitution of $r\cos\theta$ for x transforms the equation so that it can be solved, as previously explained.

The synthesizer is also applicable to the solution of a problem arising in the consideration of the tracing distortion occurring in sound reproduction from phonograph records.[2] It is desired, in the equations,

$$U=x+kar\sin kx(1+k^2a^2\sin^2 kx)^{-\frac{1}{2}}, \quad (8)$$

$$V=y+r\cos\theta=a\cos kx+r(1+k^2a^2\sin^2 kx)^{-\frac{1}{2}}, \quad (9)$$

to express V in terms of a Fourier series in U, and thereby eliminating the parameter x. This has been accomplished by expanding the parenthetical expressions in Eqs. (8) and (9), obtaining

$$U-x=kar\sin kx-\tfrac{1}{2}r(ka)^3\sin^3 kx+\tfrac{3}{8}r(ka)^5\sin^5 kx-\tfrac{5}{16}r(ka)^7\sin^7 kx+\cdots. \quad (10)$$

$$V=a\cos kx+r[1-\tfrac{1}{2}(ka)^2\sin^2 kx+\tfrac{3}{8}(ka)^4\sin^4 kx-\tfrac{5}{16}(ka)^6\sin^6 kx+\cdots]. \quad (11)$$

Choosing what were assumed to be practical values for the quantities, $k=2\pi/\lambda$, $a=$ the amplitude of the cosine curve which the stylus tip follows, $r=$ the radius of the tip of the reproducer stylus, (i.e., $k=500$, $a=0.001$ in., $r=0.002$ in.), Eqs. (10) and (11) become

$$U-x=0.000908\sin kx+0.000025\sin 3kx+0.000001\sin 5kx+\cdots. \quad (12)$$

$$V=0.002+0.001\cos kx-0.000058\cos 2kx+0.000045\cos 4kx+\cdots. \quad (13)$$

Equations (12) and (13) are represented in Fig. 8. Analysis[1] of the V *vs.* U relation obtained from these two curves gives the desired relation (V as a harmonic function of U),

$$V=51.1\cos U-3.53\cos 2U+0.55\cos 3U+1.92\cos 4U+\cdots. \quad (14)$$

Figure 9 shows the curve representing Eq. (14), and this equation shows that there is a tracing distortion of approximately 7 percent second harmonic, 1 percent third harmonic, and 4 percent fourth harmonic.

Another use to which the synthesizer may be put is that of making the double summation, as suggested by Beevers and Lipson,[3] used in the Patterson method[4] of the determination of the components of interatomic distances in crystals. The machine summation has the advantages of ease, rapidity, and a continuous representation of each variable within its range.

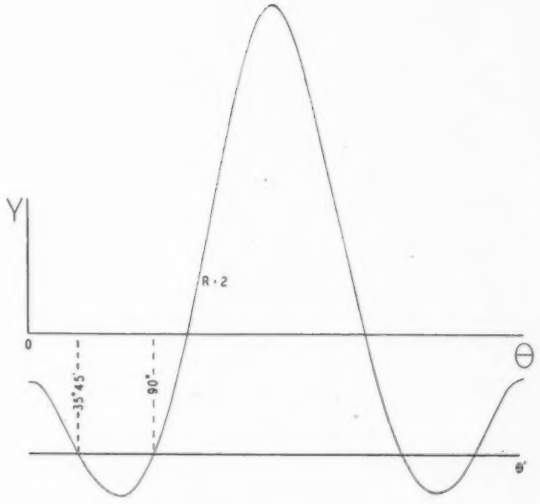

FIG. 6. $X+1=\cos HX=Y+X+1$.

The function involved is of the form

$$A(x,y)=\sum_0^n\left[\sum_0^m F^2(hk0)\cos 2\pi hx\right]\cos 2\pi ky, \quad (15)$$

where $A(x, y)$ is a weighted density distribution function of the atoms in the crystal,[4] $F(hk0)$ is the crystal structure factor, h and k are Miller indices, and x and y are fractions of the cell dimensions.

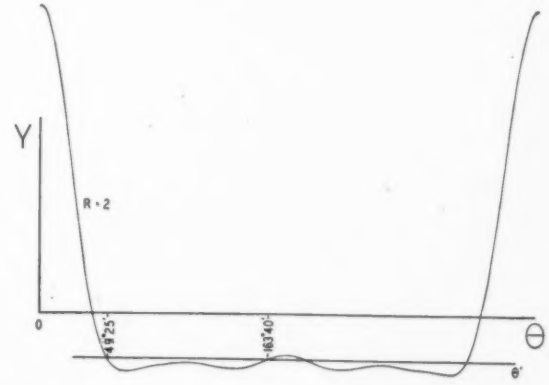

FIG. 7. $\log(2X^3+5X^2+10X+15)=5/X=Y+5/X$.

[2] J. A. Pierce, and F. V. Hunt, "On distortion in sound reproduction from phonograph records," J. Acous. Soc. Am. **534**, 14 (1938).

[3] C. A. Beevers and H. Lipson, "A rapid method for the summation of a two-dimensional Fourier series," Phil. Mag. **17**, 885 (1934).

[4] A. L. Patterson, "A direct method for the determination of the components of interatomic distances in crystals," Zeits. f. Krist. **90**, 517 (1935).

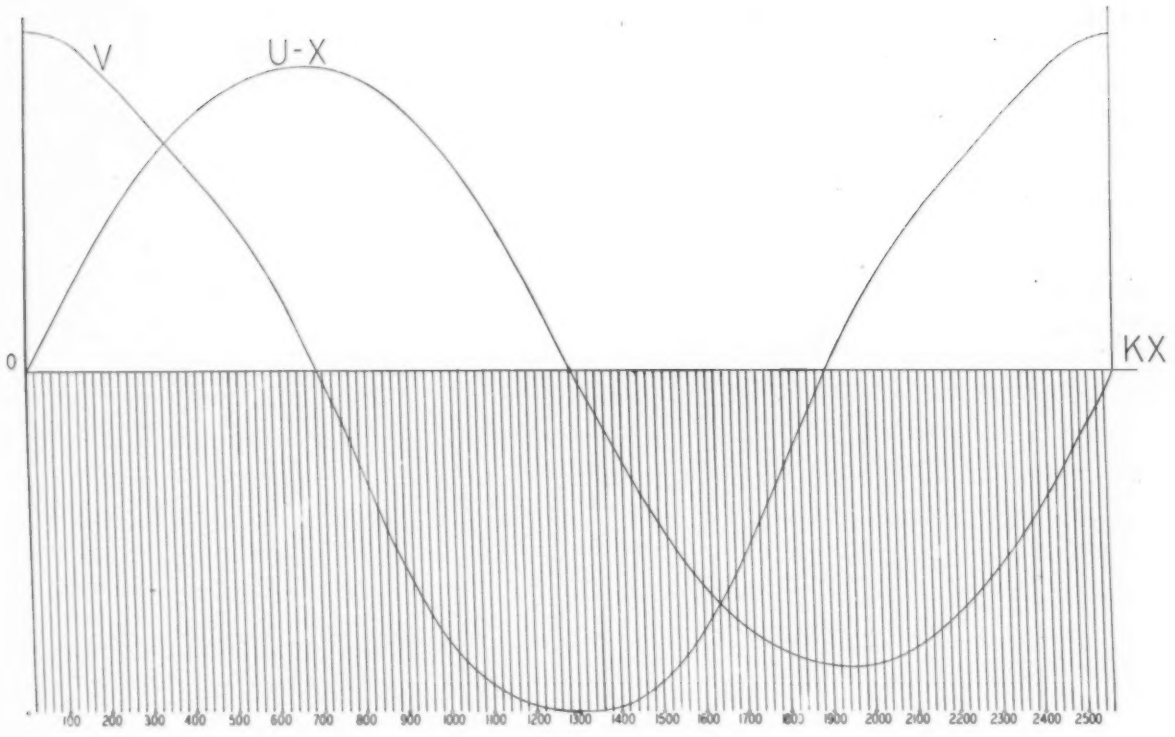

FIG. 8. $U-X=KAR \sin KX(1+K^2A^2 \sin^2 KX)^{-\frac{1}{2}}$, $V=A \cos KX+R(1+K^2A^2 \sin^2 KX)^{-\frac{1}{2}}$.

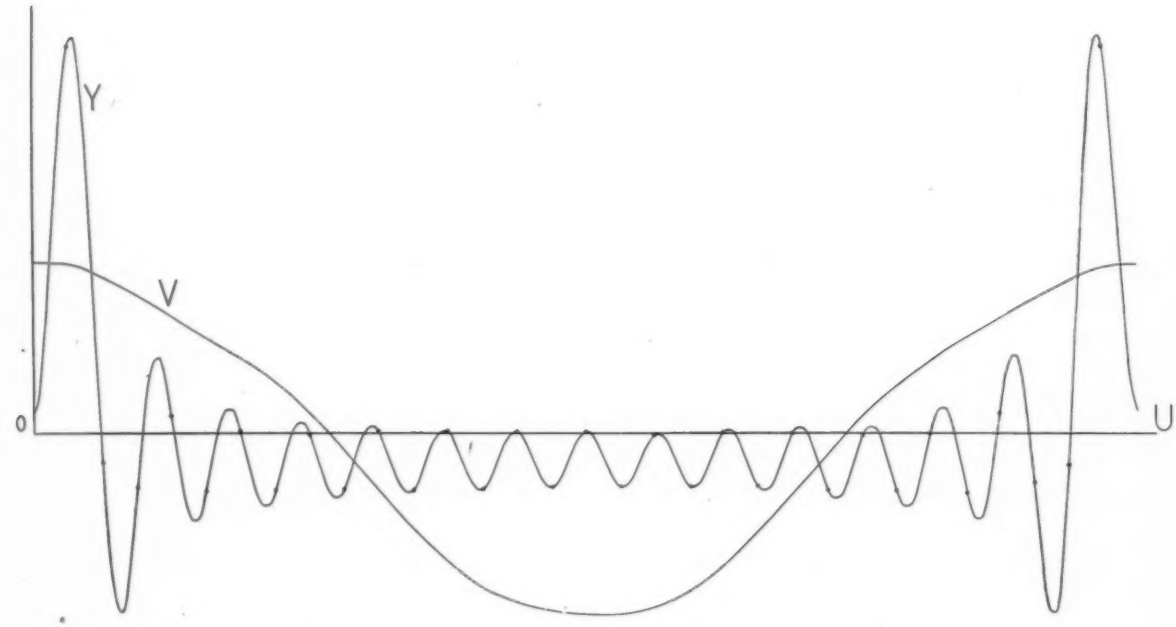

FIG. 9. $V=51.1 \cos U-3.53 \cos 2U+0.55 \cos 3U+1.92 \cos 4U$.

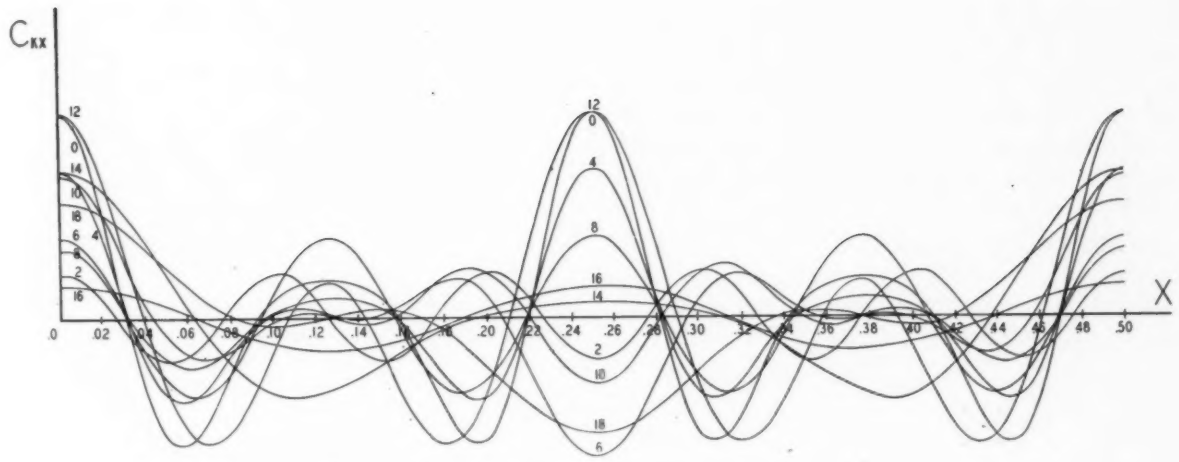

FIG. 10. Patterson summation curves.

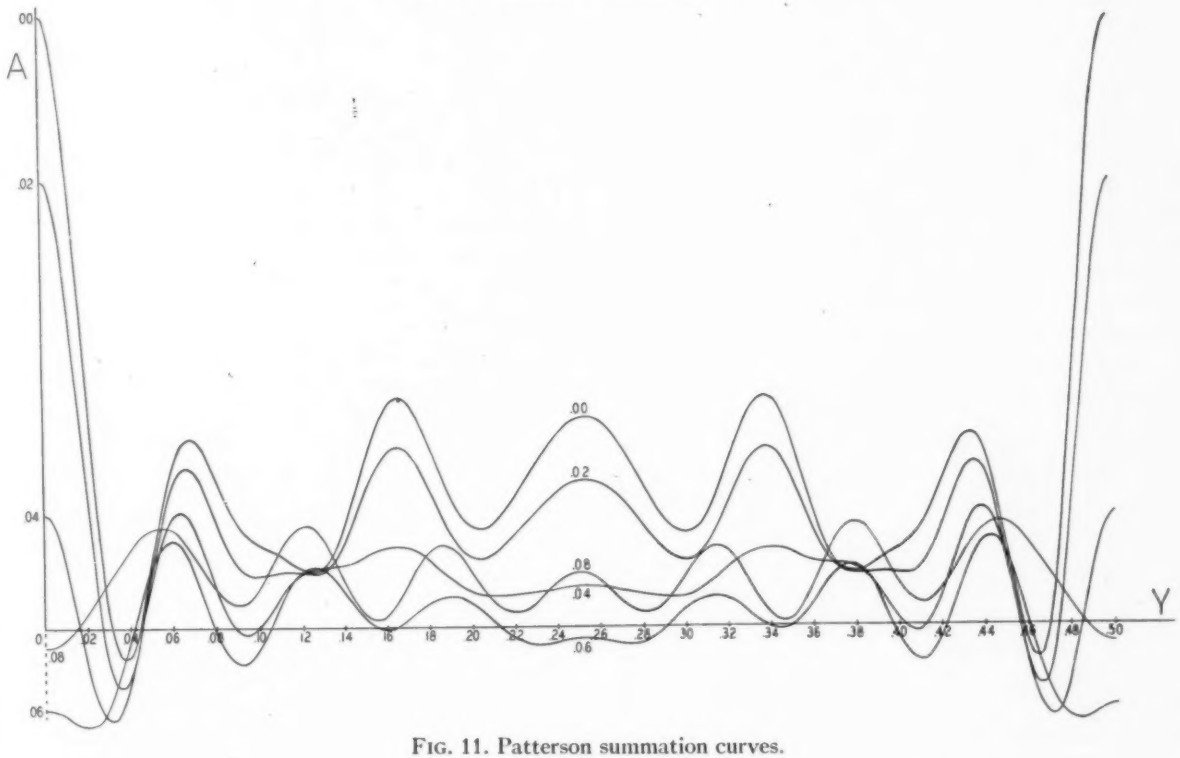

FIG. 11. Patterson summation curves.

The inside series is first evaluated. The machine angle θ is made equal to $2\pi x$, and the $F^2(hk0)$ values, after applying an appropriate scale, are the magnitudes B of the machine settings. Thus the inside series becomes

$$C_{kx}=\sum_{0}^{m} B_{hk0}\cos h\theta. \qquad (16)$$

One curve for each value of k is drawn (such a set of curves is seen in Fig. 10). The values of C_{kx} are read from the machine (or curve traced by the machine) for values of x from 0 to 1 at small intervals (e.g., at intervals of 0.02), and these values of C_{kx} are used in the second summation.

The second summation may be represented by

$$A(x, y)=\sum_{0}^{n} C_{kx}\cos 2\pi ky. \qquad (17)$$

One curve is drawn for each value of C_{kz} (i.e., C_{k0}, $C_{k.02}$, $\cdots$). Figure 11 shows such a set of curves. These final curves can be viewed as cross sections of a contour on which the displacement of the peaks from the origin (or the contour) represents the projections of the interatomic distances on an xy plane. If desired, a contour map may be drawn on an xy plane since, from the last set of curves, one can obtain the value of $A(x, y)$ for each (x, y) value. After recording the $A(x, y)$ values on the xy plane, those of the same value may be connected giving the contour lines. Similar operations may be made which give a contour map for each of the other two planes.

Throughout this paper, the curves traced by the machine are given to make clear the principles and processes involved. Actually, the curves need not be graphed since the operations are definitely facilitated by reading the ordinates from a vertical scale for the corresponding values of abscissas (angles) as read from a dial.

A Short Course in Tensor Analysis for Electrical Engineers

By Gabriel Kron. Pp. 250+xv, Figs. 152, 23½×15½ cm. John Wiley & Sons, Inc., New York, 1942. Price $4.50.

This book has been written specifically for an advanced course in Engineering at the General Electric Company, Schenectady. Part I deals with general asymmetrical networks, like multi-winding transformers, mercury-arc rectifier circuits, etc., and Part II with transient and steady state theory of rotating machinery.

The title of this book should not mislead prospective students of tensor analysis as they will find precious little "course" in this mathematical subject. Rather, a quotation of certain rules of manipulation is given, usually even with only sketchy background as to the correct application. Since the first paper of the author using tensorial vocabulary, many discussions in the literature have cleared the atmosphere of mystery surrounding the presence of tensors in stationary network theory. Though admittedly tensor notation may simplify the style of network equations, the use of a notation can hardly raise assemblages of values to the status of "physical entities." It would seem that the first part of the book will give little satisfaction to: the mathematician, on account of lack in rigor and logical foundation; the physicist, on account of lack in basic concepts and linking relations; and the engineer, on account of lack in security and proper guidance.

The second part, devoted to summary treatment of electro-mechanical rotating machine problems, is somewhat marred by the propagandistic use of the tensorial method (almost at any cost). Many of the illustrative examples demonstrate the gift of the author to see the abstract ideal form behind a maze of confusing, inessential connections. Yet, one cannot help a feeling of insecurity in using the "connection" and "transformation" tensors without verifying in some other way the correctness of the results; which, of course, mitigates against proficient use.

There are also several loose chapters involving the field equations of Maxwell.

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Glass: The Miracle Maker

By C. J. Phillips. Pp. 424+xii, Figs. 207, 16×23½ cm. Pitman Publishing Corporation, New York, 1941. Price $4.50.

The author states in his preface that this book is for those who may see in glass a unique and versatile material, interesting in its own right and full of unexplored possibilities for creating products and improving production methods.

In the opinion of the reviewer the purpose of this book has been admirably carried out. The book opens with a brief history of glass and an outline of its chemistry. The mechanical and other physical properties of glass are described in considerable detail, and effective use is made of charts and diagrams. There are chapters on the manufacture of glass and the handling of raw materials, the construction of the furnaces, and the machines used for working glass.

The second part of this book is concerned primarily with the applications of glass in the building industry, in the home, in electrical transmissions, in manufacture, and in science and research. The chapter concerned with glass in architecture seems to be especially complete. Scientists will perhaps be disappointed in the chapter on glass in science and research where applications are so numerous and so obvious that it is difficult for anyone in two dozen pages to do justice to this topic. The last chapter is concerned with Fiber Glass and will serve to inform the reader of the many new and interesting uses of glass in the form of fiber.

The book is well illustrated, and because of the many tables, charts, and diagrams will form a very useful reference book to anyone who is interested in the history, manufacture, or application of glass.

E. Hutchisson
National Defense Research Committee

Note on the Annealing of Glass

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AND

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(Received October 8, 1942)

The rate of removal of mechanical strain should follow Maxwell's first-order law, but for glasses it has been found that the rate is proportional to the square of the birefringence. The present paper explains this in terms of the structural picture of glass. During annealing the stress optical constant of the glass, as well as its viscosity, undergoes a continuous change. Birefringence, therefore, is not a correct measure of the stress, and the Adams-Williamson findings are only apparently in contradiction to the classical derivation of Maxwell.

THE annealing of optical glass involves two different processes: first, the removal of mechanical stress in order to avoid anisotropy, and second, the stabilization of the atomic structure in order to exclude secular changes in volume and refractive index. The second has usually been overlooked, but has now received the proper attention from N. Winter-Klein[1] and E. D. Tillyer.[2]

The first process, the removal of mechanical strain by viscous flow, has been generally recognized as a fundamental problem, but still offers certain difficulties to the physicist engaged in glass research. It is the aim of this note to explain these difficulties on the basis of the modern theory of the vitreous state.

Recent work on the structure of glass determined from x-ray diffraction patterns (B. E. Warren) has demonstrated that there is a close similarity between the atomic structures of vitreous and crystalline silicates. The use of optical indicators (W. A. Weyl) has made it possible to draw conclusions on the changes which occur in the atomic arrangement of oxide glasses on heating and cooling.

Crystalline silicates are affected by temperature variation in a twofold way: (1) The increasing atomic vibrations cause the crystal lattice to become more spacious. This thermal expansion is instantaneous and reversible. (2) A large number of crystals such as quartz, cristobalite, and wollastonite ($CaSiO_3$) exhibit modification (polymorphic) changes.

In the latter case, raising the temperature not only causes the interatomic distances to expand continuously with maintenance of the original structure and symmetry, but induces one or more abrupt changes in structure. Sometimes these changes are reversible, sometimes they are so sluggish that the modification can be overheated or supercooled.

The space requirements of the ions change with their thermal energy. Modification changes occur whenever the equilibrium between attraction and repulsion forces of the ions is upset by the variation of interionic distances and another arrangement suits the new energy conditions better. In many cases the high temperature form is characterized by an ion having a lower coordination number than in the low temperature form.

Glasses undergo essentially the same structural changes. Besides their reversible thermal expansion, they exhibit volume effects due to structural changes. Certain ions, for example Ni^{++} and Co^{++}, when introduced as indicators reveal by their absorption spectra that they are surrounded by six oxygens in an annealed glass, but by only four in a hot glass or in one which has been quenched.

A difference between changes of crystalline modification and corresponding structural changes in glasses is that the absence of over-all symmetry in the vitreous state allows the structural units of a glass to undergo changes over a range of temperatures. The crystalline state, on

[1] N. Winter-Klein, "Causes and effects of tempering of glass," Rev. d'optique **15**, 281–97 (1936), **16**, 361–85 (1937). "Transformation region of glass," in press.

[2] E. D. Tillyer, personal communications.

the other hand, requires cooperative maneuvers between the single units, taking place isothermally.

With glasses and some crystals there are structural changes which can be frozen in by rapid cooling. For instance, if wollastonite is melted and then quenched, pseudowollastonite is obtained. That means we have frozen in a structure which is stable at high temperatures. Different heat treatments of a calcium metasilicate melt will produce either wollastonite or pseudowollastonite. Different heat treatments of a glass melt, however, will lead to many possible atomic structures between the one stable at high temperature and the one stable at room temperature. Instead of two modifications, glasses of different thermal history represent a series of an indefinite number of individuals. Like the modifications of the crystalline silicates, these different glasses can be distinguished by their different properties, such as by density, refractive index, electroconductivity, color, or fluorescence.

This picture is nothing but an illustration of the well-known fact that all properties of glasses vary with the thermal treatment just as they vary with changes in glass composition. Birefringence, the response of the refractive properties to a mechanical strain, also is affected by the heat treatment, but this relationship has been completely overlooked in the treatment of the annealing of glass.

The removal of strain has been thoroughly discussed by Maxwell and from his mathematical treatment we learn that the rate at which the stress tends to disappear depends on amount of stress and on the viscosity or relaxation time of the body. Indeed whenever stress is introduced into a piece of glass by mechanical means, the removal of this stress by viscous flow follows Maxwell's equation. That means, at a given temperature where the glass has a stable, constant viscosity, the rate of removal depends upon the stress.

L. H. Adams and E. D. Williamson[3] made a very thorough study of the annealing of optical glass and found that the equation deduced by Maxwell did not fit the experimental facts on birefringence lowering. If the rate of release of strain were proportional to the amount of strain present, the stress should decrease exponentially with the time. Their results were described much better by another function in which the release rate is approximately proportional to the square of the stress, not to the first power.

Adams and Williamson emphasized that their equation was purely empirical and applicable only within a limited time and temperature range. The apparent contradiction between Maxwell's and Adams and Williamson's equations has produced many arguments among scientists, and some writers tried to construct theoretical reasons why glasses deviate from the first-order law (Maxwell) and follow a square root law.

H. R. Lillie[4] found that the viscosity of glass, like all other properties, changes with heat treatment. An unannealed glass has a lower viscosity than the same glass after it has been annealed. This offers a sufficient explanation why the rate of stress release should deviate from Maxwell's law. Maxwell postulates that the relaxation time, that means the viscosity, is constant during the period of stress release. In glasses this is true only when the stress is introduced by mechanical forces into an annealed specimen. Annealing an unannealed glass, however, produces a drift in viscosity.

Besides this argument contributed by the work of Lillie, we have another of equal importance. In all investigations on the annealing of glass the actual stress has been measured by the birefringence which it produces.

$$\delta = F \cdot B \cdot l.$$

The birefringence δ is a linear function of the stress F and the path length l of the light within the stressed area. B is the proportionality factor, the stress optical constant expressed in brewsters, which characterizes the glass but is independent of the absolute amount of stress.

The value of B of a quenched glass differs from that of the annealed specimen. Most glasses seem to have a higher B value when quenched, and Filon and Harris[5] found that even at room

[3] L. H. Adams and E. D. Williamson, "Annealing of glass," J. Frank. Inst. **190**, 597–631, 835–70 (1920).

[4] H. R. Lillie, "Viscosity-time-temperature relations in glass at annealing temperatures," J. Am. Ceram. Soc. **16**, 619–31 (1933).

[5] L. N. G. Filon and F. C. Harris, "Photo-elastic dispersion of vitreous silica," Proc. Roy. Soc. London, **A130**, 410–31 (1931).

temperature the stress optical constant of a glass changes and gradually approaches an equilibrium value.

This is in accord with the fact that other properties of quenched glasses undergo a similar change. From this point of view a certain amount of birefringence in a quenched glass does not indicate the same stress as the same amount of birefringence in an annealed glass under mechanical strain. Stress and birefringence are proportional only when the glass has undergone a definite heat treatment. During annealing the stress optical constant is continuously changing and it is a fallacy to identify removal of birefringence with removal of mechanical stress or to use birefringence as a quantitative measure for the transient stress present in the transforming glass.

From the fact that the removal of birefringence does not obey a first-order law, we cannot conclude that Maxwell's treatment of the removal of stress does not hold true for glass. So long as we have no quantitative information on the change of the stress optical constant with heat treatment, we have to accept the Adams-Williamson equation as an empirical equation describing the removal of birefringence. The true rate of removal of stress very probably does follow Maxwell's law, but this fact is masked because during the annealing process the relaxation time of the glass and also its stress optical factor undergo continuous change.

Erratum: The Photographic Action of Electrons in the Range Between 40 and 212 Kilovolts

[J. App. Phys. **13**, 450–456 (1942)]

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IN Fig. 5 of the above paper, "H and D curves for different accelerating voltages and different negative materials," the curves reproducing the measurements of Borries and Knoll on Agfa Isochromfilm should be shifted toward the left by unity on the axis of abscissas. The Agfa Isochromfilm, thus, is more sensitive by a factor of 10 than represented in the figure. It is the most sensitive of the three materials for which data are plotted.

A New Analytical Method for Solving van der Pol's and Certain Related Types of Non-Linear Differential Equations, Homogeneous and Non-Homogeneous

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(Received August 24, 1942)

A simple analytical treatment is developed for the differential equation $(d^2u/dt^2)-\epsilon(1-u^2)\times(du/dt)+u=0$ (van der Pol), in order to study the behavior of its solution assumed bounded in $(0, \infty)$. It is shown, without any further assumption, that, if $\epsilon \ll 1$, $u(t)$ can be approximated closely, for $t \geqq 0$, by a simple oscillation with amplitude 2. This is a more precise form of a statement due to van der Pol. It is further shown that $u^2(t)+u'^2(t) \sim 4$, $t \geqq 0$, $0<\epsilon \ll 1$. The method and results are then extended to the more general equation $(d^2u/dt^2)-\epsilon F(u)(du/dt)+u=0$, in particular, for $F(u)=1-u^4$, $1-2au-u^2$, $a=$constant, also to the non-homogeneous equation $(d^2u/dt^2)-\epsilon F(u)(du/dt)+u=$given function of t. The analytical results obtained in this paper show a remarkable agreement with those obtained for the same equations by mechanical means (on the differential analyzer).

INTRODUCTION

THE difficulty of solving non-linear differential equations lies mainly in that they possess *movable* singularities, depending upon the initial conditions. Thus, the simple non-linear equation, $y'=y^2$ has for its general solution $y=1/C-x$ which becomes infinite for $x=\xi=C$ —constant of integration determined by the initial condition: $y=y_0$ for $x=x_0$, yielding $\xi=x_0+1/y_0$.

The mobility of singularities makes it difficult to apply rigorously and effectively to such equations the ordinary methods of solution, say, that of successive approximations.

In the present paper we propose a new analytical method for solving van der Pol's celebrated differential equation of electrical oscillations[1] and more general non-linear equation of a similar type, homogeneous and non-homogeneous. This method (presented in a post-graduate course in advanced mathematics in the Moore School of Electrical Engineering, University of Pennsylvania), while not claiming to overcome fully the foregoing mathematical difficulties, is straightforward and simple and employs elementary analytical tools. It has, we believe, certain advantageous features, compared with other methods proposed.[1,2] First, we make *one assumption only*, namely: under certain initial conditions, the equation in question possesses a solution bounded for all non-negative values of time t—the independent variable. When dealing with differential equations describing physical phenomena, this assumption may be considered as justified by physical considerations or experiments. In case of van der Pol's and similar equations it is further justified in the course of our discussion (see Eq. (3) below). Aside from this assumption, we do not neglect higher derivatives or higher frequencies, as being "small," a customary procedure, incorrect, however, from the mathematical standpoint* and hardly provable experimentally. Secondly, we obtain a relation for the initial *conditions* (see Eq. (36) below) necessary for the solution to behave as specified above, and a simple relation connecting the oscillation and its derivative (see Eq. (36.1)). It is interesting to note that these relations were obtained both theoretically by the mathematician, the author, at his desk and experimentally by the engineer, members of the Moore School, on the differential analyzer, each side working independently and not knowing of the work of the other.

Last, our method is quite general and applies to a wide class of non-linear equations, homogeneous and non-homogeneous.

[1] Balth. van der Pol, "The non-linear theory of electric oscillations," Proc. I.R.E. **22**, 1051–1086 (1934).

[2] N. M. Kryloff and N. N. Bogoliouboff, *Introduction in Non-Linear Mechanics* (in Russian) (Academy of Sciences U.S.S.R., Kieff, 1937), 365 pp.; pp. 38–131.

* The function $\epsilon \sin x/\epsilon^2$, where ϵ is a small positive quantity, remains small in absolute value for all x, but this does not hold for its derivative $(1/\epsilon)\cos x/\epsilon^2$.

1. VAN DER POL'S EQUATION

We first apply our method to van der Pol's equation (reference 1, pp. 1056–1061)

$$(d^2u/dt^2)-\epsilon(1-u^2)(du/dt)+u=0, \qquad \epsilon>0. \quad (1)$$

(The equation $(d^2u/dt^2)-\epsilon(a-u^2)(du/dt)+u=0$, $a>0$, reduces to (1), by replacing u with $\sqrt{a}u$ and ϵa with ϵ.) By a "solution" of (1) we mean a function $u\equiv u(t)$ satisfying (1) which does not vanish identically. A direct attempt to solve (1) by means of a series of the form

$$u_0(t)+\epsilon u_1(t)+\epsilon^2 u_2(t)+\cdots$$

is of no avail, for it yields terms containing t outside the symbols sin, cos.

1.1 Some General Properties of Solutions of van der Pol's Equation

Equation (1) shows that the simultaneous vanishing of two of the three quantities u, $u'(=du/dt)$, $u''(=d^2u/dt^2)$ implies the vanishing of the third one for the same t. This leads to the following properties of the solutions of (1), if we make simple use of the general existence and uniqueness theorem for differential equations.

(i) *Equation (1) has no solution for which two of the quantities u, u', u'' vanish for the same (finite) value of t, for only the solution $u\equiv 0$ behaves in this manner.*

(ii) *Equation (1) has no solution u such that u or u' vanishes or oscillates (that is, crosses the t axis) infinitely many times in a finite interval.* In fact, this would imply the existence of a finite point $t=c$—cluster-point for the "zeros" (points where the function vanishes) of u and u', respectively. By continuity, $u(c)$ and $u'(c)$, respectively, vanish; so do (by the definition of the derivative) $u'(c)$ and $u''(c)$, respectively, which contradicts (*i*).

Next multiply (1) by $2u'$ and integrate. We get successively

$$(d/dt)[u^2(t)+u'^2(t)]+2\epsilon[u^2(t)-1]u'^2(t)=0, \quad (2)$$
$$t\geqq 0; \qquad \epsilon>0,$$

$$u^2(t)+u'^2(t)+2\epsilon\int_{t_0}^{t}[u^2(t)-1]u'^2(t)dt$$
$$=u^2(t_0)+u'^2(t_0), \quad 0\leqq t_0<t; \quad \epsilon>0. \quad (3)$$

The relation (3) is important for it yields further properties of solutions of (1).

(iii) *Equation (1) has no solution $u(t)$ such that*

$$\lim_{t\to T}|u(t)|=\infty, \quad (4)$$

*T finite or infinite.**

If a solution $u(t)$ exists such that $u^2(t)$ increases indefinitely, as $t\to T$ (finite), take in (3) t_0, $0<t_0<T$, so close to T that

$$u^2(t)>1, \quad t_0\leqq t<T.$$

Then the integral in (3) is non-negative; hence, with t_0 fixed,

$$u^2(t)\leqq u^2(t)+u'^2(t)<u^2(t_0)+u'^2(t_0), \quad t_0\leqq t<T, \quad (5)$$

which contradicts (4). Similar reasoning applies if in (4) $T=\infty$.

The above considerations are of a general nature and apply to differential equations of a more general type than (1). It is seen that the statements (i), (ii), (iii) do not require any assumption concerning the solutions of (1), that (i) and (ii) are valid for any real ϵ, and (iii) holds for any $\epsilon>0$. It is further seen that the integral

$$I\equiv\epsilon\int_{t_0}^{t}[u^2(t)-1]u'^2(t)dt, \quad 0\leqq t_0<t, \quad \epsilon>0,$$

acts as a "safety valve" for (1). Namely, if $u^2(t)\geqq 1$ in a certain interval (t_0, t_1), then for any t in this interval

$$u^2(t)\leqq u^2(t)+u'^2(t)<u^2(t_0)+u'^2(t_0). \quad (6)$$

Further important use of (3) will be made below.

1.2 A New Method for Solving van der Pol's Equation

We now assume that (1) possesses a solution $u(t)$, $t\geqq 0$, bounded in $(0, \infty)$. How does this solution behave? Rewrite (1) as

$$u''-\epsilon u'+u=\epsilon u^2u', \quad (u(0)=u_0, \quad u'(0)=u'_0), \quad (7)$$

where ϵ is a parameter subject to the condition

$$0<\epsilon<2, \quad (8)$$

* This property is due to the effect of the non-linear term $-\epsilon u^2u'$. In fact, the equation $u''-2u'+u=0$ has for its general solution $u=e^t(A+Bt)$, with $\lim_{t\to\infty}|u(t)|=\infty$.

and u_0, u'_0 are given constants, independent of ϵ. *The main feature of our method is the combining of (7) with the equation*

$$z''-\epsilon z'+z=0, \quad 0<\epsilon<2, \tag{9}$$

which has for its solutions

$$z_1(t)=e^{\epsilon t/2}\cos\omega t, \quad z_2(t)=e^{\epsilon t/2}\sin\omega t, \quad \omega=(1-\epsilon^2/4)^{\frac{1}{2}}. \tag{10}$$

Let

$$w(t)=z(t)u'(t)-z'(t)u(t), \tag{11}$$

with

$$w'(t)=z(t)u''(t)-z''(t)u(t). \tag{12}$$

Combining (7) and (9) yields for $w\equiv w(t)$ a linear differential equation of the first order

$$w'-\epsilon w=-\epsilon u^2u'z,$$

whence, by an elementary formula,

$$we^{-\epsilon t}=w_0-\epsilon\int_0^t e^{-\epsilon x}u^2(x)z(x)u'(x)dx, \quad t\geqq 0 \quad (w_0\equiv w(0)).$$

Substituting here $z_{1,2}(t)$ for $z(t)$ gives

$$w_1(t)e^{-\epsilon t}=w_{01}-(\epsilon/3)\int_0^t e^{-\epsilon x}z_1(x)u^2(x)u'(x)dx, \tag{13}$$

$$w_2(t)e^{-\epsilon t}=w_{02}-(\epsilon/3)\int_0^t e^{-\epsilon x}z_2(x)u^2(x)u'(x)dx,$$

where

$$w_i=z_iu'_i-z'_iu_i, \quad w_{0i}=w_i(0), \quad i=1,2. \tag{14}$$

Substituting (14) in (13) yields an algebraic system of two linear equations, with unknown $u(t)$, $u'(t)$, and the solution of (1) is obtained by solving this system for $u(t)$. We have, by (10), (11), (12), (14),

$$z_2w_1-z_1w_2=-u(z'_1z_2-z_2'z_1)=\omega ue^{\epsilon t}, \tag{15}$$

$$z_{01}=1, \quad z_{02}=0, \quad z'_{01}=\epsilon/2, \quad z'_{02}=\omega, \quad w_{01}=u'_0-\epsilon u_0/2, \quad w_{02}=-\omega u_0, \tag{16}$$

and solving the system (13) for $u(t)$ we get

$$u(t)\omega=z_2w_{01}-\epsilon\int_0^t e^{-\epsilon x}z_1(x)u^2(x)u'(x)dx - z_1w_{02}-\epsilon\int_0^t e^{-\epsilon x}z_2(x)u^2(x)u'(x)dx. \tag{17}$$

Integrating by parts on the right gives:

$$u(t)\omega=z_2\Big[u'_0-\epsilon u_0/2+\epsilon u_0^3/3 +(\epsilon/3)\int_0^t e^{-\epsilon x}u^3(x)\Big(\frac{dz_1}{dx}-\epsilon z_1(x)\Big)dx -z_1\Big[-\omega u_0+(\epsilon/3)\int_0^t e^{-\epsilon x}u^3(x) \times\Big(\frac{dz_2}{dx}-\epsilon z_2(x)\Big)dx. \tag{18}$$

Here we replace $dz_{1,2}/dx$ by simple expressions derived from (10), and we let

$$\int_0^t e^{-\epsilon x/2}u^3(x)\cos\omega x dx=-\alpha_1-e^{-\epsilon t/2}A_1(t), \quad \int_0^t e^{-\epsilon x/2}u^3(x)\sin\omega x dx=-\alpha_2-e^{-\epsilon t/2}A_2(t), \tag{19}$$

where

$$-\alpha_1=\int_0^\infty e^{-\epsilon x/2}u^3(x)\cos\omega x dx, \quad -\alpha_2=\int_0^\infty e^{-\epsilon x/2}u^3(x)\sin\omega x dx, \quad e^{-\epsilon t/2}A_1(t)=\int_t^\infty e^{-\epsilon x/2}u^3(x)\cos\omega x dx, \quad e^{-\epsilon t/2}A_2(t)=\int_0^\infty e^{-\epsilon x/2}u^3(x)\sin\omega x dx, \tag{20}$$

$$A_1(0)=-\alpha_1, \quad A_2(0)=-\alpha_2.$$

The integrals in (20) exist, since $u(t)$ is bounded in $(0, \infty)$. Note that $A_i(t)$, $A'_i(t)$, $i=1, 2$, are also bounded in $(0, \infty)$. In fact, if $|u(t)|\leqq H$, $t\geqq 0$ (H, fixed constant, independent of t), then we conclude from (20) that

$$|A_i(t)|\leqq H^3e^{\epsilon t/2}\int_t^\infty e^{-\epsilon t/2}dt=2H^3/\epsilon, \quad |A'_i(t)|\leqq 2H^3, \quad t\geqq 0; \quad i=1,2. \tag{21}$$

Equation (18) now yields

$$u(t)=\frac{\sin \omega t}{\omega}\{e^{\epsilon t/2}[u'_0-\epsilon u_0/2 + \epsilon u_0^3/3+\epsilon^2\alpha_1/6+\epsilon\omega\alpha_2/3] + \epsilon^2 A_1(t)/6+\epsilon\omega A_2(t)/3\} + \frac{\cos \omega t}{\omega}\{e^{\epsilon t/2}[\omega u_0+\epsilon\omega\alpha_1/3-\epsilon^2\alpha_2/6] + \epsilon\omega A_1(t)/3-\epsilon^2 A_2(t)/6\}, \quad t\geqq 0. \quad (22)$$

We now come to an important point in our discussion. By (22), *in order that $u(t)$ shall be bounded in $(0, \infty)$ the terms involving $e^{\epsilon t/2}$ must vanish.* Thus, the unknown constants $\alpha_{1,2}$ must satisfy the following equations:

$$\epsilon^2\alpha_1/6+\epsilon\omega\alpha_2/3+\epsilon u_0^3/3-\epsilon u_0/2+u'_0=0, \qquad \epsilon\omega\alpha_1/3-\epsilon^2\alpha_2/6+\omega u_0=0. \quad (23)$$

This system can be always solved for $\alpha_{1,2}$, its determinant being $-\epsilon^2/9\neq 0$. We get

$$a_1=3\epsilon u_0/2-\epsilon u'_0/2-3u'_0/2+3u_0/\epsilon, \qquad \alpha_2=3\omega u_0-\omega u_0^3-3\omega u'_0/\epsilon. \quad (24)$$

With such $\alpha_{1,2}$, (22) becomes

$$u(t)=\frac{\sin \omega t}{\omega}\left[\frac{\epsilon^2 A_1(t)}{6}+\frac{\epsilon\omega A_2(t)}{3}\right] + \frac{\cos \omega t}{\omega}\left[\frac{\epsilon\omega A_1(t)}{3}-\frac{\epsilon^2 A_2(t)}{6}\right]. \quad (25)$$

In view of (21), (25) shows that $u'(t)$ *is also bounded in* $(0, \infty)$; so is, by (1), $u''(t)$, and so are, again by (21), $A''_{1,2}(t)$.

Furthermore, (25) throws some light on the above constant

$$H=\max_{(0,\infty)}|u(t)|. \quad (26)$$

The identity

$$(a_1b_1+a_2b_2)^2=(a_1^2+a_2^2)(b_1^2+b_2^2)-(a_1b_2-a_2b_1)^2,$$

valid for any real $a_{1,2}$, $b_{1,2}$, yields the inequality

$$|a_1b_1+a_2b_2|\leqq[(a_1^2+a_2^2)(b_1^2+b_2^2)]^{\frac{1}{2}}.$$

This, applied to (25), gives

$$|u(t)|\leqq(\epsilon/3\omega)[A_1^2(t)+A_2^2(t)]^{\frac{1}{2}}, \quad t\geqq 0. \quad (27)$$

$|u(t)|$ attains its maximum H for some value $t=t_0$. Letting $t=t_0$ in (27) readily gives

$$H\equiv\max_{(0,\infty)}|u(t)|\geqq(3.2^{\frac{1}{2}}\cdot\omega)^{\frac{1}{2}}/2, \quad 0<\epsilon<2. \quad (28)$$

We proceed to transform (25). Substitute therein

$$A_i(t)=B_i(t)-\alpha_i, \quad i=1, 2, \ (B_i(0)=0), \quad (29)$$

and make use of (23). We get

$$u(t)=(u'_0-\epsilon u_0/2+\epsilon u_0^3/3)(\sin \omega t/\omega) + u_0\cos \omega t+[\epsilon^2 B_1(t)/6-\epsilon\omega B_2(t)/3] \times(\sin \omega t/\omega)+[(\epsilon\omega B_1(t)/3) -(\epsilon^2 B_2(t)/6)]\cos \omega t/\omega. \quad (30)$$

Here

$$B_{1,2}(t), \quad B'_{1,2}(t), \quad B''_{1,2}(t)$$

are all bounded in $(0, \infty)$. Thus we may write finally

$$u(t)=(u'_0-\epsilon u_0/2+\epsilon u_0^3/3)(\sin \omega t/\omega) + u_0\cos \omega t+\epsilon P(t), \quad (31)$$

$$u(t)=(u'_0/\omega)\sin \omega t+u_0\cos \omega t+\epsilon Q(t), \quad (31.1)$$

where $P(t)$, $Q(t)$, $P'(t)$, $Q'(t)$, $P''(t)$, $Q''(t)$ are all bounded in $(0, \infty)$ (these bounds generally depending on ϵ).

Note that *our considerations and the results obtained do not require ϵ to be very small ($\epsilon\ll 1$), but only the inequality $0<\epsilon<2$.*

Formulae (31), (31.1) evidently hold also for $\epsilon=0$, for they give the precise expressions for $u(t)$, $u'(t)$, $u''(t)$, if $u(t)$ is the solution of

$$u''+u=0, \quad (u(0)=u_0, \quad u'(0)=u'_0),$$

obtained by letting $\epsilon=0$ in (1). We thus conclude that *in (31) and (31.1) the quantities* $|\epsilon P(t)|$, $|\epsilon P'(t)|$, $|\epsilon P''(t)|$, $|\epsilon Q(t)|$, $|\epsilon Q'(t)|$, $|\epsilon Q''(t)|$ *are very small, for $t\geqq 0$, if $\epsilon\ll 1$.* With such ϵ we can go further in our discussion.

1.3. Case of $0<\epsilon\ll 1$

Formulae (31), (31.1) show that here our solution $u(t)$, $t\geqq 0$, is approximated by a simple oscillation whose frequency and amplitude are,

respectively,

$$\omega=(1-\epsilon^2/4)^{\frac{1}{2}}\sim 1-\epsilon^2/8,$$
$$M=[(u'_0-\epsilon u_0/2+\epsilon u_0^3/3)^2\cdot\omega^{-2}+u_0^2]^{\frac{1}{2}} \tag{32}$$
$$\sim(u_0^2+u'^2_0)^{\frac{1}{2}}\equiv A.$$

This is to be compared with van der Pol's result (reference 1, pp. 1057–1061), which gives for the frequency the value unity—that of the limiting case $\epsilon=0$.* Rewrite (31.1) as

$$u(t)=A\sin(\omega t+\alpha)+\epsilon Q(t),\quad \alpha=\text{const.}, \tag{33}$$

and substitute in (3), transformed into

$$\frac{u^2(t)+u'^2(t)}{2\epsilon\int_{t_0}^{t}u'^2(t)dt}+\frac{\int_{t_0}^{t}u^2(t)u'^2(t)dt}{\int_{t_0}^{t}u'^2(t)dt}$$
$$=1+\frac{u^2(t_0)+u'^2(t_0)}{2\epsilon\int_{t_0}^{t}u'^2(t)dt},\quad 0\leqq t_0<t. \tag{34}$$

For the sake of brevity, denote generally by $B(t)$, not necessarily the same in different expressions, a function such that $B(t)$, $B'(t)$, $B''(t)$ are bounded in $(0,\ \infty)$ and $|\epsilon B(t)|$, $|\epsilon B'(t)|$, $|\epsilon B''(t)|$ are very small, $t\geqq 0$ if $\epsilon\ll 1$. We write generally

$$\epsilon B(t)=\eta(t),\quad |\eta(t)|\leqq\eta\ll 1,\quad t\geqq 0.$$

It follows that

$$\int_{t_0}^{t}\eta(t)dt=t\eta(t),\quad t>t_0\geqq 0.$$

We have now

$$u^2(t)=(A^2/2)(1+\cos 2\gamma)+\epsilon B(t),$$
$$\gamma=\omega t+\alpha$$
$$u'^2(t)=(A^2\omega^2/2)(1-\cos 2\gamma)+\epsilon B(t),$$
$$\int_{t_0}^{t}u'^2(t)dt=A^2\omega^2t/2+B(t)+t\eta(t),$$
$$0\leqq t_0<t$$
$$\int_{t_0}^{t}u^2(t)u'^2(t)dt=A^4\omega^2t/8+B(t)+t\eta(t).$$

Equation (34) now becomes

$$\frac{B(t)}{2\epsilon[A^2\omega^2t/2+B(t)+t\eta(t)]}+\frac{A^4\omega^2t/8+B(t)+t\eta(t)}{A^2\omega^2t/2+B(t)+t\eta(t)}$$
$$=1+\frac{u^2(t_0)+u'^2(t_0)}{2\epsilon[A^2\omega^2t/2+B(t)+t\eta(t)]}. \tag{34.1}$$

Moreover, for t very large and $0<\epsilon\ll 1$, it is seen that

$$A^4\omega^2t/8+B(t)+t\eta(t)\sim A^4\omega^2t/8,$$

and the extreme terms on either side of (34.1) become very small. Thus, (34.1) leads to

$$A^2/4+\eta(t)\sim 1,\quad t\to\infty,\quad 0<\epsilon\ll 1,$$

and since A is independent of t,

$$A^2/4\sim 1,\quad A\sim 2\quad (0<\epsilon\ll 1). \tag{35}$$

Equation (33) can now be rewritten as

$$u(t)\sim 2\sin(\omega t+\alpha),\quad t\geqq 0,\quad 0<\epsilon\ll 1. \tag{33.1}$$

Thus, *for small ϵ, the oscillation $u(t)$ under discussion may be approximated, for $t\geqq 0$, by a simple oscillation with a fixed* amplitude 2 (the closeness of such approximation depending on the smallness of ϵ).* Equation (35), in view of (32), gives further

$$u_0^2+u_0'^2\sim 4,\quad (0<\epsilon\ll 1), \tag{36}$$

as a necessary condition for the solution $u(t)$ of (1) to behave in $(0,\ \infty)$ as specified above.

Now observe that (1) *does not contain t explicitly*, so that replacing t by $t-t_0$, $t_0>0$, arbitrary, brings the origin to t_0, *without changing* (1). It follows that in (36) we can replace $t=0$ by any positive t. Hence, the following important property of the solution $u(t)$ under discussion:

$$u^2(t)+u'^2(t)\sim 4,\quad t\geqq 0,\quad (0<\epsilon\ll 1). \tag{36.1}$$

As stated in the Introduction, (36.1) was obtained experimentally.[3]

* Kryloff and Bogolioubоff (reference 2, p. 1047), by an entirely different method based upon the use of Fourier series, obtain for the frequency $1-\epsilon^2/16$.

* See references 1 and 2, where a less precise statement is made, namely, that $u(t)$ approaches, as $t\to\infty$, a stationary oscillation with amplitude 2.

[3] J. G. Brainerd and C. N. Weygandt, "Unsymmetrical self-excited oscillations in certain simple non-linear systems," Proc. I.R.E. **24**, 914–922 (1936), p. 920.

2. GENERALIZATION

The foregoing method for solving van der Pol's equation applies to the more general type

$$u''-\epsilon F(u)+u'=0, \tag{37}$$

where $F(u)$ is a given function of u bounded in any finite interval, for example, *$F(u)$ is a polynomial*, with coefficients independent of ϵ. Again we assume that, under certain initial conditions, (37) has a solution $u(t)$, $t\geqq 0$, bounded in $(0, \infty)$.

Introduce once more Eq. (9) and develop, with very slight changes, the considerations of Section 1.2, rewriting (37), at the suggestion of Professor C. N. Weygandt, as

$$u''-\epsilon[1-f(u)]u'+u=0, \quad 0<\epsilon<2 \tag{38}$$
$$(u(0)=u_0, \quad u'(0)=u'_0).$$

Here $f(u)$ is a given function of u bounded in any finite interval, so that $f[u(t)]$, as a function of t, is bounded in $(0, \infty)$. (Thus, van der Pol's Eq. (1) is a special case of (38), with $f(u)\equiv u^2/3$.)

It suffices to give the final results quite similar to the foregoing.

Formula (25) holds, with the following significance of $\alpha_{1,2}$, $A_{1,2}(t)$:

$$3\int_0^t e^{-\epsilon x/2}f(u)\cos\omega x dx=-\alpha_1-e^{-\epsilon t/2}A_1(t),$$
$$3\int_0^t e^{-\epsilon x/2}f(u)\sin\omega x dx=-\alpha_2-e^{-\epsilon t/2}A_2(t), \tag{39}$$

$$-\alpha_1=3\int_0^\infty e^{-\epsilon x/2}f(u)\cos\omega x dx,$$
$$-\alpha_2=3\int_0^\infty e^{-\epsilon x/2}f(u)\sin\omega x dx, \tag{40}$$

$$e^{-\epsilon t/2}A_1(t)=3\int_t^\infty e^{-\epsilon x/2}f(u)\cos\omega x dx,$$
$$e^{-\epsilon t/2}A_2(t)=3\int_t^\infty e^{-\epsilon x/2}f(u)\sin\omega x dx. \tag{41}$$

The unknown constants $\alpha_{1,2}$ satisfy the following equations:

$$\epsilon^2\alpha_1/6+\epsilon\omega\alpha_2/3+u'_0+\epsilon u_0/2+\epsilon f_0=0,$$
$$\epsilon\omega\alpha_1/3-\epsilon^2\alpha_2/6+\omega u_0=0 \tag{42}$$
$$(f_0\equiv f(0)).$$

Formula (31) is replaced with

$$u(t)=(u'_0-\epsilon u_0/2+\epsilon f_0)(\sin\omega t/\omega)$$
$$+u_0\cos\omega t+\epsilon P(t), \quad t\geqq 0. \tag{43}$$

Hence, (31.1), holds, with A as in (32), namely:

$$u(t)=A\sin(\omega t+\alpha)+\epsilon Q(t), \quad t\geqq 0 \tag{43.1}$$
$$A=[u_0^2+u_0'^2]^{\frac{1}{2}},$$

where $Q(t)$, $Q'(t)$, $Q''(t)$ are bounded in $(0, \infty)$ (these bounds generally depending on ϵ), and $|\epsilon Q(t)|$, $|\epsilon Q'(t)|$, $|\epsilon Q''(t)|$ are small for $t\geqq 0$, if $\epsilon\ll 1$.

Regarding the results of Section 1.1, we draw the following conclusions.

(i) *Equation (37) has no solution $u(t)$ such that u and u', or u' and u'', or, in case $F(0)\neq 0$, u and u'', vanish for the same finite value of t.*

(ii) *Statement* (ii) *of Section 1.1 holds for (37) without change.*

(iii) *Statement* (iii) *of Section 1.1 holds, if $F(u)$ is negative for u sufficiently large, in particular, if $F(u)$ is a polynomial of even degree with highest coefficient <0.* This is derived by the above reasoning from (3), where $u^2(t)-1$ is replaced with $-F(t)$.

As to the results, properly modified, of Section 1.3, they also hold for certain types of $F(u)$. We give here two illustrations.

1°. Consider (38) in the form

$$u''-\epsilon(1-u^4)u'+u=0, \quad 0<\epsilon<2. \tag{44}$$

Here all three statements (i), (ii), (iii) of Section 1.1 hold. Equation (3) is here replaced with

$$u^2(t)+u'^2(t)+2\epsilon\int_{t_0}^t(u^4-1)u'^2dt=u^2(t_0)+u'^2(t_0),$$
$$\epsilon>0, \quad 0\leqq t_0<t.$$

Rewriting this last relation as

$$\frac{u^2(t)+u'^2(t)}{2\epsilon\int_{t_0}^{t}u'^2(t)dt}+\frac{\int_{t_0}^{t}u^4u'^2dt}{\int_{t_0}^{t}u'^2(t)dt}=1+\frac{u^2(t_0)+u'^2(t_0)}{2\epsilon\int_{t_0}^{t}u'^2(t)dt}, \quad (45)$$

and making use of (43.1), we can repeat the reasoning employed above in connection with (34) and (34.1), observing that

$$u^4(t)=(A^4/8)(3+4\cos 2\gamma+\cos 4\gamma)+\epsilon B(t), \qquad \gamma=\omega t+\alpha$$

$$\int_{t_0}^{t}u^4(t)u'^2(t)dt=(A^6\omega^2t/16)+B(t)+t\eta(t), \qquad 0\leqq t_0<t.$$

The final result is $A^4\sim 8 (0<\epsilon\ll 1)$, so that, in analogy with (35), (36), and (36.1),

$$A\sim 8^{\frac{1}{4}}, \quad u^2(t)+u'^2(t)\sim 8^{\frac{1}{4}}, \quad t\geqq 0, \quad 0<\epsilon\ll 1. \quad (46)$$

2°. For a second illustration take (38) in the form (reference 3, p. 915)

$$u''-\epsilon(1-2au-u^2)u'+u=0, \quad 0<\epsilon<2, \quad (47)$$
a – constant independent of ϵ.

Here again (i), (ii), and (iii) of Section 1.1 remain valid. We have further

$$\frac{u^2(t)+u'^2(t)}{2\epsilon\int_{t_0}^{t}u'^2(t)dt}+\frac{\int_{t_0}^{t}[u^2(t)+2au(t)]u'^2(t)dt}{\int_{t_0}^{t}u'^2(t)dt}=1+\frac{u^2(t_0)+u'^2(t_0)}{2\epsilon\int_{t_0}^{t}u'^2(t)dt}, \quad 0\leqq t_0<t, \quad \epsilon>0.$$

If $\epsilon\ll 1$, $u(t)$ has the form (43.1). We may repeat the reasoning applied to (45), making use of the relation

$$2au(t)u'^2(t)=A^3\omega^2a(\sin\gamma-\sin\gamma\cos 2\gamma)+\epsilon B(t),$$

so that, for t very large and $\epsilon\ll 1$

$$\frac{\int_{t_0}^{t}2auu'^2dt}{\int_{t_0}^{t}u'^2dt}=\frac{A^3\omega^2a[B(t)+t\eta(t)]}{A^2\omega^2t/2+B(t)+t\eta(t)}\sim\eta(t), \qquad 0\leqq t_0<t.$$

We conclude that *here (35) and (36.1) hold without change*, and the additional term $\epsilon\cdot 2auu'$ in (47) has no effect. Thus, for (44) and (47) we have, respectively,

$$u(t)=8^{\frac{1}{4}}\sin(\omega t+\alpha)+\epsilon B(t), \quad u(t)=2\sin(\omega t+\alpha)+\epsilon B(t). \quad t\geqq 0.$$

In other words, *if $0<\epsilon\ll 1$, the oscillation represented by (44) or (47) may be approximated, for $t\geqq 0$, by a simple oscillation with a fixed amplitude $8^{\frac{1}{4}}$ and 2*, respectively, in full analogy to van der Pol's equation.

For Eq. (47), the relation (36.1) has been also obtained experimentally (reference 3, p. 917).

It is easy to extend the foregoing results to the case of $F(u)$ in (37) being a given polynomial, of degree n, say.

3. NON-HOMOGENEOUS EQUATIONS

We close the discussion with an application of our method to a non-linear non-homogeneous equation of the type considered which we write as

$$u''-\epsilon(1-f'(u))u'+u=\varphi'(t), \quad 0<\epsilon<2 \qquad (u(0)=u_0, \quad u'(0)=u'_0). \quad (48)$$

Here $f(u)$ and $\varphi(t)$ are given functions, $f(u)$, as a function of u, is bounded in any finite interval, $\varphi(t)$ is bounded in $(0, \infty)$. It follows that $f[u(t)]$, as a function of t is bounded in $(0, \infty)$, since we assume that (48) possesses a solution $u(t)$, $t\geqq 0$, bounded in $(0, \infty)$. Rewrite (48) as

$$u''-\epsilon u'+u=-[\epsilon f'(u)-\varphi'(t)], \quad 0<\epsilon<2,$$

and again make use of (9) in the foregoing manner. It suffices to give a few first and last steps.

We get (see (11)–(14))

$$w_i e^{-\epsilon t} = w_{0i} - \int_0^t e^{-\epsilon t} z_i(t)[\epsilon f'(u)u' - \varphi'(t)]dt,$$

$$w_i e^{-\epsilon t} = w_{0i} - (\epsilon f_0 - \varphi_0) z_{0i} + \int_0^t [\epsilon f(u) - \varphi(t)] \cdot (d/dt)[e^{-\epsilon t} z_i(t)]dt, \quad i = 1, 2 \ (\varphi_0 \equiv \varphi(0)).$$

Expanding and solving for $u(t)$ (see (18)–(20), (22)–(25)), we get

$$\begin{aligned} u(t) = (\sin \omega t/\omega)\{ & e^{\epsilon t/2}[u'_0 - \epsilon u_0/2 + \epsilon f_0 - \varphi_0 \\ & + \epsilon^2\alpha_1/6 + \epsilon\omega\alpha_2/3 - \epsilon c_1/2 - \omega c_2] \\ & + \epsilon^2 A_1(t)/6 + \epsilon\omega A_2(t)/3 - \epsilon C_1(t)/2 \\ & - \omega C_2(t)\} + (\cos \omega t/\omega)[\omega u_0 + \epsilon\omega\alpha_1/3 \\ & - \epsilon^2\alpha_2/6 + \epsilon c_2/2 - \omega c_1] + \epsilon\omega A_1(t)/3 \\ & - \epsilon^2 A_2(t)/6 + \epsilon C_2(t)/2 - \omega C_1(t)\}. \end{aligned} \quad (49)$$

Here $\alpha_{1,2}$, $A_{1,2}(t)$ are the same as in (40) and (41), respectively, and

$$\begin{aligned} -c_1 &= \int_0^\infty e^{-\epsilon t/2}\varphi(t) \cos \omega t dt, \\ -c_2 &= \int_0^\infty e^{-\epsilon t/2}\varphi(t) \sin \omega t dt, \\ e^{-\epsilon t/2} C_1(t) &= \int_t^\infty e^{-\epsilon t/2}\varphi(t) \cos \omega t dt, \\ e^{-\epsilon t/2} C_2(t) &= \int_t^\infty e^{-\epsilon t/2}\varphi(t) \sin \omega t dt. \end{aligned} \quad (50)$$

$A_{1,2}(t)$, $C_{1,2}(t)$ and their derivatives are bounded in $(0, \infty)$. The unknown constants $\alpha_{1,2}$ satisfy the following equations:

$$\begin{aligned} \epsilon^2\alpha_1/6 + \epsilon\omega\alpha_2/3 + u'_0 - \epsilon u_0/3 \\ + \epsilon f_0 - \varphi_0 - \epsilon c_1/2 - \omega c_2 = 0, \\ \epsilon\omega\alpha_1/3 - \epsilon^2\alpha_2/6 + \omega u_0 + \epsilon c_2/2 - \omega c_1 = 0, \end{aligned} \quad (51)$$

always solvable. Equation (49) now yields

$$\begin{aligned} u(t) = (\sin \omega t/\omega)[& (\epsilon^2 A_1(t)/6 + (\epsilon\omega A_2(t)/3 \\ & - (\epsilon C_1(t)/2) - \omega C_2(t)] + (\cos \omega t/\omega) \\ & \cdot [(\epsilon\omega A_1(t)/3) - (\epsilon^2 A_2(t)/6) \\ & + (\epsilon C_2(t)/2) - \omega C_1(t)], \quad t \geqq 0. \end{aligned}$$

Hence, $u'(t)$ is bounded in $(0, \infty)$.

$$\begin{aligned} u(t) = (u'_0 - \epsilon u_0/2 + \epsilon f_0 - \varphi_0 - \epsilon c_1/2 - \omega c_2) \\ \cdot (\sin \omega t/\omega) + (\omega u_0 + \epsilon c_2/2 - \omega c_1) \\ \cdot (\cos \omega t/\omega) + (\sin \omega t/\omega)\{\epsilon^2 B_1(t)/6 \\ + \epsilon\omega B_2(t)/3 - \epsilon C_1(t)/2 - \omega C_2(t)\} \\ + (\cos \omega t/\omega)\{\epsilon\omega B_1(t)/3 - \epsilon^2 B_2(t)/6 \\ + \epsilon C_2(t)/2 - \omega C_1(t)\}, \quad t \geqq 0, \end{aligned} \quad (52)$$

$$B_i(t) = A_i(t) - \alpha_i, \quad i = 1, 2.$$

$$\begin{aligned} u(t) &= M \sin(\omega t + \alpha) + \epsilon P(t) \\ &\quad - C_2(t) \sin \omega t - C_1(t) \cos \omega t, \\ M^2 &= (u'_0 - \epsilon u_0/2 + \epsilon f_0 - \varphi_0 - \epsilon c_1/2 - \omega c_2)\omega^{-2} \\ &\quad + (\omega u_0 + \epsilon c_2/2 - \omega c_1)^2, \end{aligned} \quad (53)$$

$$M \sim [(u'_0 - c_2^2 - \varphi_0)^2 + (u_0 - c_1)^2]^{\frac{1}{2}}, \quad \epsilon \ll 1.$$

Let $\epsilon = 0$ in (48), that is, consider the equation

$$u'' + u = \varphi'(t) \quad (u(0) = u_0, \quad u'(0) = u'_0).$$

Its well-known solution is

$$\begin{aligned} u(t) = (u'_0 - \varphi_0) \sin t + u_0 \cos t \\ + \cos t \int_0^t \varphi(x) \cos x dx + \sin t \int_0^t \varphi(x) \sin x dx. \end{aligned}$$

Assume

$$\int_t^\infty e^{-\epsilon x/2}\varphi(x) \begin{matrix} \sin \\ \cos \end{matrix} \omega x dx, \quad t \geqq 0, \quad (54)$$

converges uniformly with respect to ϵ, $0 \leqq \epsilon < \delta$, δ arbitrarily small but fixed. Note that (54) implies the existence of

$$\int_t^\infty \varphi(x) \begin{matrix} \sin \\ \cos \end{matrix} x dx, \quad t \geqq 0.$$

Then $u(t)$ can be put in the following form:

$$\begin{aligned} u(t) = & \left(u'_0 - \varphi_0 + \int_0^\infty \varphi(x) \sin x dx\right) \sin t \\ & + \left(u_0 + \int_0^\infty \varphi(x) \cos x dx\right) \cos t \\ & - \sin t \int_t^\infty \varphi(x) \sin x dx \\ & - \cos t \int_t^\infty \varphi(x) \cos x dx. \end{aligned}$$

This last expression, by (50), is precisely what becomes of (53), if we set there $\epsilon=0$. Hence, under (54), formula (53) holds also for $\epsilon=0$. We conclude that, *under (54), $|\epsilon P(t)|$ in (53) is very small for $t\geqq 0$, if $\epsilon\ll 1$.*

Keeping (54), we can go further. Let

$$\psi(t)=\int_t^\infty \varphi(x)\sin\omega x dx.$$

Then

$$-\int_t^\infty e^{-\epsilon x/2}\varphi(x)\sin\omega x dx=\int_t^\infty e^{-\epsilon x/2}\psi'(x)dx$$

$$=-e^{-\epsilon t/2}\psi(t)+(\epsilon/2)\int_t^\infty e^{-\epsilon x/2}\psi(x)dx$$

$$=e^{-\epsilon t/2}[-\psi(t)+\psi(\tau)],\quad \tau\geqq t.$$

It follows (see (50)) that

$$C_2(t)=-\psi(t)+\psi(\tau)\to 0,\quad \text{as}\quad t\to\infty,$$

and similarly for $C_1(t)$. We thus conclude that *under condition (54)*: (i) $\lim_{t\to\infty} C_{1,2}(t)=0$; (ii) *(from (53)). The oscillation $u(t)$ under discussion represented by (48), where $0<\epsilon\leqq 1$, approaches, for t very large, a stationary state given by*

$$u(t)\sim B\sin(\omega t+\beta),\qquad (55)$$

$$B=[u'_0{}^2-\varphi_0-c_2)^2+(u_0-c_1)^2]^{\frac{1}{2}},\quad \beta=const.$$

Observe that *(54) is satisfied and the above conclusions hold if in (48)*

$$\varphi(t)=Ke^{-ht}\sin(Et+l),\ K, h, l, E\ constants,\qquad (56)$$

$$K\neq 0,\quad h>0.$$

With such $\varphi(t)$ we can go still further. Corresponding to (48), an equation analogous to (3) can be written in the form

$$\frac{u^2(t)+u'^2(t)}{2\epsilon\int_0^t u'^2(t)dt}+\frac{\int_0^t f'(u)u'^2(t)dt}{\int_0^t u'^2(t)dt}=1$$

$$+\frac{u_0{}^2+u'_0{}^2}{2\epsilon\int_0^t u'^2(t)dt}+\frac{\int_0^t \varphi'(t)u'(t)dt}{2\epsilon\int_0^t u'^2(t)dt},\qquad (57)$$

$$t>0,\quad \epsilon>0.$$

Furthermore,

$$\int_0^t \varphi'(t)u'(t)dt=\varphi'(t)u(t)-\varphi'_0u'_0-\int_0^t \varphi''(t)u(t)dt.$$

Here, if $0<\epsilon\ll 1$ and $t\to\infty$, the integral on the left remains bounded, in view of (55), so that the last term on the right side of (57) tends to zero. In other words, *here we can treat (57) as if we were dealing with the homogeneous case.* Recalling the results of Section 2, we conclude that *the oscillation $u(t)$ represented by either of the two equations*

$$u''-\epsilon(1-u^4)u'+u=Ke^{-ht}\sin(Et+l),\qquad (58.1)$$

$$u''-\epsilon(1-2au-u^2)u'+u=Ke^{-ht}\sin(Et+l),\qquad (58.2)$$

where $0<\epsilon\ll 1$ and K, h, E, a, l are constants, $K\neq 0$, $h>0$, reaches, for t very large, a stationary state represented respectively by

$$u\sim 8^{\frac{1}{2}}\sin(\omega t+\beta),\qquad (59.1)$$

$$(\beta=\text{const.})$$

$$u\sim 2\sin(\omega t+\beta).\qquad (59.2)$$

Moreover, for t very large, we have, respectively,

$$u^2(t)+u'^2(t)\sim 8^{\frac{1}{2}},\qquad (60.1)$$

$$u^2(t)+u'^2(t)\sim 4.\qquad (60.2)$$

The last conclusion follows from the fact that although the right-hand terms in Eqs. (58.1) and (58.2) depend explicitly on t, still, replacing t with $t-t_0$ (t_0 chosen arbitrarily) does not change their form.

Finally observe that (53) holds if $\varphi(t)$ has the form (56) with $h=0$.

Two New Hot-Filament Resistances

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This report gives the results of attempts to produce a gaseous atmosphere in which the thermal conductivity increases with increasing ambient temperature much more rapidly than the normal, thereby causing a hot filament mounted in such an atmosphere to behave as if it had a negative temperature coefficient of resistance. There is also described a means of producing a gaseous atmosphere the thermal conductivity of which *decreases* very rapidly as the ambient temperature is increased, so that a hot filament mounted in such an atmosphere behaves as though it had a positive temperature coefficient of resistance three to four times as great as normal.

RECENT experiments have shown that it is possible to mount a conductor of the first class, having a positive temperature coefficient of resistance, in such a manner that the conductor behaves as if it had a *negative* temperature coefficient of about the same order of magnitude as the positive coefficient of copper. Further, by a slight modification of conditions, this same conductor can be made to exhibit a positive temperature coefficient of resistance three or four times as great as the normal for that conductor. These striking examples of anomolous behavior are not the result of accidental discoveries, but can be predicted from a knowledge of how the thermal conductivity of a confined gaseous atmosphere affects the loss of heat from a hot filament mounted in that atmosphere.

It is well known that the temperature of a hot filament does not increase as rapidly as the temperature of the confined body of gas surrounding the filament when the external temperature is increased. It can be readily observed, for example, that the (E/I) resistance of commercial ballast tubes is practically independent of ambient temperature. Owing to the increase in thermal conductivity of the gaseous atmosphere as the ambient temperature is increased, the filament, which has a positive temperature coefficient of resistance, behaves as though it had only a very small positive coefficient or none at all.

Starting from zero pressure, a hot filament would lose heat more rapidly by conduction as the pressure of the gas surrounding it is increased up to the point at which the conductivity becomes independent of pressure. It naturally follows that any means of increasing the number of gaseous molecules confined in a sealed tube as the ambient temperature is increased would cause a hot filament mounted in that atmosphere to lose heat more rapidly as the ambient temperature is increased; and the filament would exhibit a negative temperature coefficient of resistance.

Accordingly, it can be predicted that a hot filament mounted in a tube, such as shown diagramatically in Fig. 1, would exhibit a

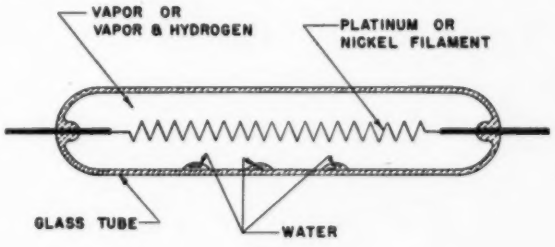

FIG. 1.

negative temperature coefficient of resistance as the ambient temperature is increased, if the atmosphere surrounding the filament consisted solely of vapor molecules of a not-too-volatile liquid, and a small quantity of the liquid adhered to the inner wall of the tube. This prediction has been completely verified by experiment, and the results have been, in the case of water, about what would be expected from the approximate relative conductivity curve shown in Fig. 2. This curve can be only approximate, since, in addition to the increase in the concentration of water vapor molecules as the ambient temperature is increased, the high temperature coefficient of thermal conductivity of water vapor would also

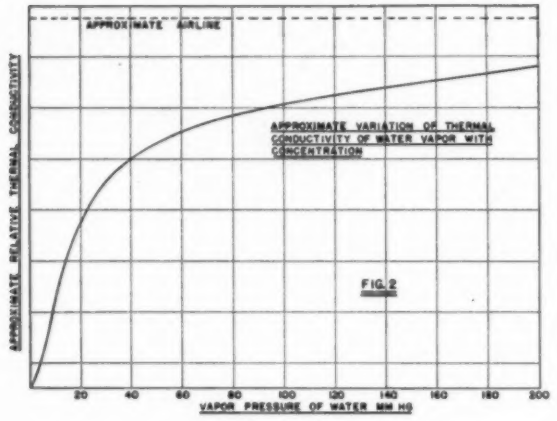

FIG. 2.

play an important part in increasing the thermal conductivity of the atmosphere surrounding the wire.

As an example of what can be expected, the following figures are given for one of the experimental tubes. At 0°C the (E/I) resistance of a certain nickel wire mounted in a tube containing only water vapor and a little water was 42.4 ohms when carrying 0.135 amp. At 28° the resistance was 38.2 ohms, while at 56° the resistance was 35.2 ohms.

All that is necessary to produce an atmosphere the thermal conductivity of which decreases very rapidly with temperature is the addition of hydrogen to the tube described above. The atmosphere then contains a gas of high thermal conductivity and a vapor having a low thermal conductivity. As the ambient temperature is increased more of the water in the tube is vaporized, thereby increasing the concentration of the water vapor in the vapor-hydrogen mixture, the thermal conductivity of which varies with the concentration of water vapor approximately as shown in Fig. 3. It can be readily understood what would happen to a hot filament mounted in such an atmosphere as the ambient temperature is increased. Since the thermal conductivity of the atmosphere would decrease very rapidly with temperature, the resistance of the heated filament would increase much more rapidly with ambient temperature than normally observed.

Typical of the behavior of this type of tube containing a nickel filament, hydrogen at 30-mm pressure, and a little water is one which had a (E/I) resistance of 24.7 ohms at 0°C. At 35° the resistance was 38.7 ohms, which is more than three times the increase in resistance which would normally have been obtained with this particular specimen of nickel.

These resistors are easily made. The most important part of the procedure is to make sure that air is removed from the tubes as completely as possible. Water is put into the tube before attaching to the pump, and helps to remove the air during the pumping; but the tube should be surrounded with a freezing mixture before all the water evaporates. It is necessary to leave only a drop or two in a tube of moderate volume. The tube should be as large as practicable so as to have as much space as possible between the filament and the wall. In the case where hydrogen is added, this operation is performed after all the air is removed. Other liquids besides water can

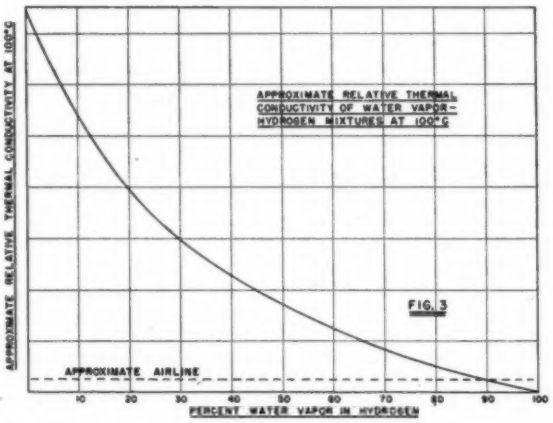

FIG. 3.

be used if the filament temperature is not too high. However, water is the most stable liquid available, and will not act on platinum, or on nickel, if the temperature of the filament is not too high.

The two resistances described above should be useful in many laboratory circuits. It should be noted, however, that they are hot-wire resistors, and are of no value unless there is enough current flowing in the circuit to heat the filament to a sufficiently high temperature above surroundings.

Errata: Theory of Vehicle Suspensions

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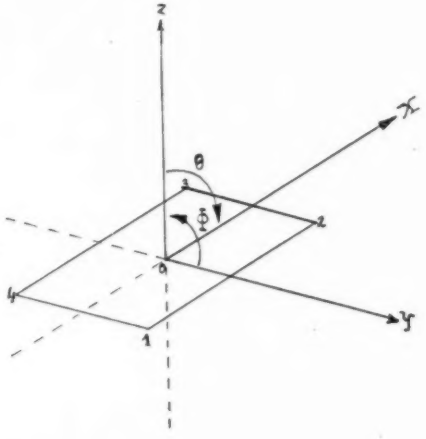
Page 485:

Delete lines three through seven below Fig. 1, beginning "It will be noted. . . ."
See new Fig. 1 below.



Page 486:

Read as follows:

First column, line three: z_1 instead of $-z_1$.
First column, line seven: z_1'' instead of $-z_1''$.
First column, line 24: delete "since z is positive downwards."
Figure 2: read z_1'' instead of $-z_1''$ and z_1 instead of $-z_1$.
Equation (3): read $\Phi_0=(z_1+z_2-z_3-z_4)/4T$ where $2T=$ track.
Equation (4): read $\theta_0=(z_1-z_2-z_3+z_4)/4W$ where $2W=$ wheelbase.
Equation (5): read $\psi_0=(z_1-z_2+z_3-z_4)/4T$.
First Eq. (6): read z'_0 instead of z'_1.

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Figure 3: read U instead of B under the arrow.
Equation (7): Delete minus sign after the equal sign.
Equation (8): For left-hand term, read F_1 instead of $-F_1$.
Equation (10): For left-hand term, read F_1 instead of $-F_1$.

Equation (11): read

$$-\xi=-16F=dV/dz_0''$$
$$-\lambda=-16RT^2=dL/d\Phi_0''$$
$$-\mu=-16PW^2=dM/d\theta_0''$$
$$-\nu=-16HT^2=2dN/d\psi_0''.$$

In last line of second column, read sprung mass instead of whole suspension.

Page 488:

For Eq. (13), read:

$$z_0(t+2W/u)-z_0(t)=-W[\theta_0(t)+\theta_0(t+2W/u)],$$
$$\Phi_0(t+2W/u)-\Phi_0(t)=-\tfrac{1}{2}[\psi_0(t)+\psi_0(t+2W/u)].$$

Equation (14), read:

$$z_0+z_0'+T(\Phi_0+\Phi_0'+\psi_0+\psi_0')+W(\theta_0+\theta_0').$$

Equation (15), read:

$$F_1=\tfrac{1}{4}\xi z_0''+\tfrac{1}{4}\left(\frac{\lambda}{T}\Phi_0''+\frac{\mu}{W}\theta_0''+\frac{\nu}{T}\psi_0''\right).$$

Equation (16), read:

$$-K(z_0'+T(\Phi_0'+\psi_0')+W\theta_0').$$

Equation (17a), read:

$$\frac{d^2}{dt^2}[z_0+z_0'+T(\phi_0+\phi_0'+\psi_0+\psi_0')+W(\theta_0+\theta_0')]$$
$$=\frac{1}{m}\left\{\frac{\xi}{4}z_0''+\frac{1}{4}\left(\frac{\lambda}{T}\phi_0''+\frac{\mu}{W}\theta_0''+\frac{\nu}{T}\psi_0''\right)-K(z_0'+T(\phi_0'+\psi_0')+W\theta_0')\right\}.$$

Page 490:

For Eq. (24a), read:

$$\frac{d^2\theta_0''}{dt^2}+\left\{\mu\left(\frac{1}{4mW^2}+\frac{1}{\rho^2M}\right)+\frac{K}{m}\right\}\theta_0''-\frac{K}{m}\theta_0'''=-\frac{K}{m}\theta_0.$$